Magnetoconductance Autocorrelation Function for Few–Channel Chaotic Microstructures

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Abstract

Using the Landauer formula and a random matrix model, we investigate the autocorrelation function of the conductance versus magnetic

field strength for ballistic electron transport through few-channel microstructures with the shape of a classically chaotic billiard coupled to ideal leads. This function depends on the total number M of channels and the parameter t which measures the difference in magnetic field strengths. Using the supersymmetry technique, we calculate for any value of M the leading terms of the asymptotic expansion for small t. We pay particular attention to the evaluation of the boundary terms. For small values of M, we supplement this analytical study by a numerical simulation. We compare our results with the squared Lorentzian suggested by semiclassical theory and valid for large M. For small M, we present evidence for non–analytic behavior of the autocorrelation function at $t = 0$.

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Recent advances in semiconductor technology have made it possible to study experimentally the transport of ballistic electrons through microstructures [1, 2] with the shape of a classically chaotic billiard. Two aspects of the dependence of the conductance $G(B) = (e^2/h)g(B)$ on an external magnetic field B have received [3, 4] particular attention: The suppression of weak localization at small values of B, and the form of the conductance autocorrelation function $C(\Delta B) = \overline{\delta g(B)\delta g(B + \Delta B)}$. Here, $\delta g(B) = g(B) - \overline{g(B)}$ is the difference between the dimensionless conductance $g(B)$ and its mean value. The bars indicate an average which experimentally is taken over the Fermi energy or the applied gate voltage.

In a previous paper [5], we have calculated the generic form of the weak localization peak. In the present paper, we address the generic form of the conductance autocorrelation function C. For ideal coupling between leads and microstructure, this function depends on the number M of channels in the leads, and on t, a measure of ΔB to be defined below.

Several authors have calculated the autocorrelation function C in an approximate way. In the semiclassical approximation [6], C was found to have the form of a squared Lorentzian. The semiclassical limit is expected to apply only when $M \gg 1$. Moreover, the non–diagonal terms in the Gutzwiller trace formula are neglected. Attempts to determine C from a random matrix model for the scattering matrix with the help of a Brownian motion model [7, 8] have not been successful. Use of a random matrix model and the supersymmetry technique [9, 10] has confirmed the squared Lorentzian shape of C. However, these were also asymptotic calculations valid in the limit $M \gg 1$ and thus kin to the semiclassical approximation.

In the actual experiments [3, 4], the number of channels per lead was quite small. Thus, it is not clear whether the theoretical results just mentioned apply to this situation. Moreover, a theoretical study of the full autocorrelation function is of considerable interest in its own right. Indeed, C is a four–point function, and to the best of our knowledge it has not been possible so far to calculate this kind of function without approximation analytically, using a random matrix model as starting point. The work reported in this paper does not solve this problem fully either. However, we succeed in calculating analytically the leading terms in an asymptotic expansion of C for small t. We believe that in so doing, we display interesting and novel aspects of the supersymmetry technique, especially in relation to the boundary terms. In addition, we are led to question whether C is analytic in t near $t = 0$.

Starting point is the random matrix model for the Hamiltonians of the chaotic microstructure for two values of the external field. Our approach is very close to that taken in Refs. [9, 10], and also to our work on weak localization [5], except that we now have to deal with a four–point function. We use the Landauer formula and Efetov's supersymmetry technique [11] in the form of Ref. [12].

The calculation of the leading terms of the asymptotic expansion of the autocorrelation function for small t involves a generalization of the usual polar coordinates used for two–point functions. The transformation to these variables necessarily introduces the boundary or Efetov–Wegner terms into the formalism. We devote particular attention to these terms. To the best of our knowledge, integrals in polar coordinates over graded vectors have first been treated by Parisi and Sourlas [13], and over graded matrices by Efetov [11] and Wegner, see Ref. [12]. Our central result given in Eq. (74) below has first been derived in a different way by Efetov [11] and by Zirnbauer [14]. The derivation which we employ makes use of the method of boundary functions suggested by Berezin [15]. For a more general discussion of coordinate transformations in graded integrals we refer to the papers by Rothstein [16], and by Zirnbauer and Haldane [17].

As a complement to the analytical work which yields only the leading terms of the asymptotic expansion, we simulate the correlation function numerically. We focus attention on small values of the channel number M.

In Section 1, we formulate our approach. The integral theorem used in calculating the asymptotic expansion, is derived in Section 2. The analytical calculation is discussed in Section 3, our numerical results are presented in Section 4. Section 5 contains the conclusions. Additional mathematical details are given in two Appendices.

1 Formulation of the Problem

Much of the development in this section is similar to that of ref. [5] where further details may be found. We try to keep this presentation both brief and self–contained.

1.1 Physical Assumptions

We thus consider ballistic electron transport through a two–dimensional microstructure with the shape of a classically chaotic billiard in the presence of an external magnetic field B. We wish to determine the generic features of the autocorrelation function

$$\overline{\delta g(B^{(1)})\delta g(B^{(2)})} = \overline{g(B^{(1)})g(B^{(2)})} - \overline{g(B^{(1)})}\ \overline{g(B^{(2)})}. \tag{1}$$

Here, $g(B)$ is the dimensionless conductance, and the bar denotes an average. Experimentally, this average is typically taken over the applied gate voltage [3], while theoretically we take it over a suitably chosen random matrix model. We assume that the Hamiltonians $H^{(r)}$ of the closed billiard exposed to the two fields $B^{(r)}$, $r = 1, 2$, are described by two ensembles of Hermitean random matrices $H^{(r)}_{\mu\nu}$ of dimension N which have the form

$$H^{(1)}_{\mu\nu} = H^{(I)}_{\mu\nu} - \sqrt{\frac{t}{N}}H^{(II)}_{\mu\nu}, \quad H^{(2)}_{\mu\nu} = H^{(I)}_{\mu\nu} + \sqrt{\frac{t}{N}}H^{(II)}_{\mu\nu}. \tag{2}$$

The matrices $H^{(C)}_{\mu\nu}$ with $C = I, II$ belong to two independent (uncorrelated) Gaussian unitary ensembles (GUE's) with first and second moments given by

$$\overline{H^{(C)}_{\mu\nu}} = 0, \quad \overline{H^{(C)}_{\mu\nu}H^{(C)}_{\mu'\nu'}} = \frac{\lambda^2}{N}\delta_{\mu\nu'}\delta_{\mu'\nu}. \tag{3}$$

The parameter λ fixes the mean level spacing d of each ensemble and must be adjusted to the local mean level spacing at the Fermi energy E_F of the billiard. In the limit $N \to \infty$ always considered in this paper, the average level density d^{-1} takes the shape of Wigner's semicircle, and the Fermi energy $E_F = 0$ is chosen to lie at its center where $d = \pi\lambda/N$. As shown in ref. [5], the parameter t is related to the area A of the billiard by $\sqrt{t} = k|B^{(1)} - B^{(2)}|A/(2\phi_0)$ where k is a numerical constant of order unity and where $\phi_0 = hc/e$ is the elementary flux quantum.

For this model to be valid, the fields $B^{(r)}$ must obey the following two conditions. (i) $B^{(r)}$ must be strong enough so that for both $r = 1, 2$, the crossover transition from orthogonal to unitary symmetry is complete. In practice this requires fields stronger than a few millitesla, cf. Ref. [5]. (ii) The fields must be weak enough for a GUE description to be valid. In other

words, the classical cyclotron radius must be large compared to the diameter of the billiard.

The billiard is attached to two quasi one–dimensional leads. In each lead, there are $M/2$ transverse modes which define the incoming and outgoing channels. These channels are labelled $a = 1, \ldots, M/2$ in lead one and $b = M/2+1, \ldots, M$ in lead two. With $|\mu>$ denoting a complete orthonormal set of basis states in the billiard, and c the channels in either lead, we denote the coupling matrix elements between billiard and leads by $W_{c\mu}(E)$. We assume that these matrix elements obey time–reversal symmetry, $W_{c\mu}(E) = W^*_{c\mu}(E)$ so that the effect of the magnetic field is confined entirely to the interior of the billiard. The characteristic dependence on energy E of the elements $W_{c\mu}(E)$ is typically very slow on the scale of the mean level spacing d, and is therefore neglected.

We write $g(B^{(r)}) = g^{(r)}$ and use for $g^{(r)}$, $r = 1, 2$, the Landauer formula for zero temperature,

$$g^{(r)} = \sum_{a=1}^{M/2} \sum_{b=M/2+1}^{M} \left(|S^{(r)}_{ab}|^2 + |S^{(r)}_{ba}|^2 \right) . \tag{4}$$

The elements of the scattering matrix $S^{(r)}_{cd}$ are given by

$$S^{(r)}_{cd} = \delta_{cd} - 2i\pi \sum_{\mu\nu} W_{c\mu} [D^{(r)}]^{-1}_{\mu\nu} W_{d\nu} \tag{5}$$

where $D^{(r)}$ is the inverse propagator

$$D^{(r)}_{\mu\nu} = E\delta_{\mu\nu} - H^{(r)}_{\mu\nu} + i\pi \sum_c W_{c\mu} W_{c\nu} . \tag{6}$$

We must put $E = E_F$ and recall that $E_F = 0$. For $a \neq b$, we have

$$|S^{(r)}_{ab}|^2 = 4 \text{ Tr} \left(\Omega^a [D^{(r)}]^{-1} \Omega^b [D^{(r)\dagger}]^{-1} \right) , \tag{7}$$

where the trace runs over the level index μ, and where

$$\Omega^c_{\mu\nu} = \pi W_{c\mu} W_{c\nu} . \tag{8}$$

For later use, we also define

$$\Omega_{\mu\nu} = \sum_c \Omega^c_{\mu\nu} = \sum_c \pi W_{c\mu} W_{c\nu} . \tag{9}$$

Because of the unitary symmetry of our matrix ensemble, the coupling matrix elements $W_{c\mu}$ appear in all ensemble averages only in the form of the unitary invariants $X_{cd} = \pi \sum_\mu W_{c\mu} W_{d\mu}$. As in ref. [5], we assume that $X_{cd} = X_c \, \delta_{cd}$ is diagonal in the channel indices. The parameters X_c appear in the final expressions only via the "transmission coefficients" or "sticking probabilities" T_c given by

$$T_c = 4\lambda X_c (\lambda + X_c)^{-2} \ . \tag{10}$$

With $0 \leq T_c \leq 1$, the transmission coefficients measure the strength of the coupling between billiard and leads. Motivated by the absence of barriers between billiard and leads, we put $T_c = 1$ for all c (maximal coupling). Our model then depends only on M and on $|B^{(1)} - B^{(2)}|$, i.e., on two parameters which are fixed by experiment. All dependence on E_F and on λ disappears (the parameter λ appears only in the transmission coefficients T_c).

The ensemble average of $g^{(r)}$ is obviously the same as for the case of a single GUE, $\overline{g^{(r)}} = M/2$ for $r = 1, 2$. The calculation of the autocorrelation function, Eq. (1), therefore reduces to that of the ensemble average $\overline{g^{(1)} g^{(2)}}$. To find $\overline{g^{(1)} g^{(2)}}$, we use both analytical and numerical methods. Our analytical study of the autocorrelation function presented in this and in the following three sections is based on Efetov's supersymmetry method [11] in the version of ref. [12]. We refer to the latter paper as VWZ. The numerical calculation of the autocorrelation function is discussed in Section 4.

1.2 Supersymmetry Approach

We deal with a four–point function. This necessitates some changes in the formulation of VWZ.

The generating function $Z(J)$ is given by the graded Gaussian integral

$$Z(J) = \int \mathcal{D}[\Phi] \exp \left(i \sum_{\alpha r p} < \overline{\Phi}_{\alpha r p}, [(D + J)\Phi]_{\alpha r p} > \right) \tag{11}$$

where we use the notation

$$< F, G >= \sum_\mu F(\mu) G(\mu). \tag{12}$$

The $8N$ dimensional graded vector Φ has components $\Phi_{\alpha rp}(\mu)$. The supersymmetry index α distinguishes ordinary complex (commuting) integration variables ($\alpha = 0$) and Grassmannn (anticommuting) variables ($\alpha = 1$). Later, the two values 0 and 1 of the supersymmetry index α will also be denoted by b and f, respectively. As in subsection 1.1, the index $r = 1, 2$ refers to the two fields. The index $p = 1, 2$ refers to the retarded and the advanced propagator as usual. Finally, $\mu = 1, \ldots, N$. The adjoint graded vector is defined by $\overline{\Phi}(\mu) = \Phi^\dagger(\mu)s$, with

$$s_{\alpha rp,\alpha' r' p'} = (-)^{(1+\alpha)(1+p)} \; \delta_{\alpha\alpha'}\delta_{rr'}\delta_{pp'} \; . \tag{13}$$

The $8N$ dimensional graded matrix D is the graded inverse propagator,

$$D = E \; \mathbf{1}_8 \times \mathbf{1}_N - H + iL \times \Omega \; , \tag{14}$$

with

$$H = \mathbf{1}_8 \times H^{(I)} - \tau_3 \times \sqrt{\frac{t}{N}} H^{(II)} \; . \tag{15}$$

The $8N$ dimensional graded matrices L and τ_3 are given by

$$L_{\alpha rp,\alpha' r' p'} = (-)^{(1+p)} \; \delta_{\alpha\alpha'}\delta_{rr'}\delta_{pp'}, \; [\tau_3]_{\alpha rp,\alpha' r' p'} = (-)^{(1+r)} \; \delta_{\alpha\alpha'}\delta_{rr'}\delta_{pp'} \; . \tag{16}$$

The $8N$ dimensional graded matrix $J = J(\epsilon)$, a function of M^2 ordinary variables $\epsilon^{(rp)}_{ab}$, has the form

$$J(\epsilon) = \sum_{rp} I^{(rp)} \times J^{(rp)}(\epsilon) \; , \tag{17}$$

where

$$I^{(rp)}_{\alpha' r' p',\alpha'' r'' p''} = (-)^{(1+\alpha')} \; \delta_{rr'}\delta_{pp'}\delta_{\alpha'\alpha''}\delta_{r'r''}\delta_{p'p''} \; , \tag{18}$$

$$J^{(rp)}_{\mu'\mu''}(\epsilon) = \pi \sum_{ab} W_{a\mu'} W_{b\mu''} \epsilon^{(rp)}_{ba} \; . \tag{19}$$

The integration measure is $\mathcal{D}[\Phi] = \prod_{\alpha rp\mu} \mathrm{d}\Phi_{\alpha rp}(\mu)\mathrm{d}\Phi^*_{\alpha rp}(\mu)$. Performing the Gaussian integration, we obtain

$$Z(J) = \mathrm{Detg}^{-1}(D + J) \tag{20}$$

where the symbol Detg stands for the graded determinant. Eqs. (14) to (19) show that the matrix $D + J$ consists of four diagonal blocks corresponding to

the two different magnetic fields and to retarded and advanced propagators, respectively. The function $Z(J)$ factorizes accordingly. Differentiating the resulting expression with respect to the source parameters ϵ and averaging over the ensemble yields

$$\overline{|S^{(1)}_{ab}|^2|S^{(2)}_{a'b'}|^2} = \frac{\partial^4}{\partial\epsilon^{(11)}_{ab}\partial\epsilon^{(12)}_{ba}\partial\epsilon^{(21)}_{a'b'}\partial\epsilon^{(22)}_{b'a'}}\overline{Z(J)}|_{\epsilon=0} \ . \tag{21}$$

For later developments, we note that $Z(J)$ possesses an important symmetry. We consider transformations $T : \Phi(\mu) \to T\Phi(\mu)$ which leave the bilinear form $\overline{\Phi}(\mu)\Phi(\mu)$ invariant. Such transformations satisfy the condition

$$T^{-1} = sT^{\dagger}s \ . \tag{22}$$

The set of 8 dimensional matrices T forms a graded unitary group $G = \mathrm{U}(2,2/4)$ with compact and non–compact bosonic subgroups. For $t = \Omega = J = 0$, the generating function $Z(J)$ is invariant under this group.

1.3 Saddle–Point Approximation

We average $Z(J)$ over the ensemble, apply the Hubbard–Stratonovich transformation as usual to reduce the number of independent integration variables, and use the limit $N \to \infty$ to define a saddle–point approximation.

In Eq. (11), we have to average the term

$$\exp\left(-i\sum_{\alpha r p} < \overline{\Phi}_{\alpha r p}, (H\Phi)_{\alpha r p} >\right) \ . \tag{23}$$

Using Eq. (2), we find that the average is the product of two terms,

$$\overline{\exp\left(-i\sum_{\alpha r p} < \overline{\Phi}_{\alpha r p}, (\mathbf{1}_8 \times H^{(I)}\Phi)_{\alpha r p} >\right)} = \exp\left(-\frac{\lambda^2}{2N}\mathrm{trg}S^2\right) \ , \tag{24}$$

$$\overline{\exp\left(i\sum_{\alpha r p} < \overline{\Phi}_{\alpha r p}, (\tau_3 \times \sqrt{\frac{t}{N}}H^{(II)}\Phi)_{\alpha r p} >\right)} = \exp\left(-\frac{t}{N}\frac{\lambda^2}{2N}\mathrm{trg}[(S\tau_3)^2]\right) \ , \tag{25}$$

where

$$S_{\alpha r p,\alpha' r' p'} =< \Phi_{\alpha r p}, \overline{\Phi}_{\alpha' r' p'} >= \sum_{\mu} \Phi_{\alpha r p}(\mu)\overline{\Phi}_{\alpha' r' p'}(\mu) \ . \tag{26}$$

Collecting results, we have with $E = E_F = 0$,

$$\overline{Z(J)} = \int \mathcal{D}[\Phi] \exp\left(i\mathcal{L}(S) + i \sum_{\alpha r p} < \overline{\Phi}_{\alpha r p}, [(iL \times \Omega + J)\Phi]_{\alpha r p} > \right) \ , \quad (27)$$

where

$$i\mathcal{L}(S) = -\frac{\lambda^2}{2N}(\mathrm{trg} S^2 + \frac{t}{N}\mathrm{trg}[(S\tau_3)^2]) \ . \quad (28)$$

The following steps are quite standard and are only sketched here. After the Hubbard–Stratonovich transformation and integration over the variables Φ, $\overline{Z(J)}$ is written as an integral over the eight–dimensional graded matrices σ,

$$\begin{aligned} \overline{Z(J)} = \int \mathcal{D}[\sigma] \exp\left(-\frac{N}{2}(\mathrm{trg}\ \sigma^2 + \frac{t}{N}\mathrm{trg}[(\sigma\tau_3)^2]) \right) \times \\ \exp\left(-\mathrm{Trg}\ \ln\ [-\lambda(\sigma + \frac{t}{N}\tau_3\sigma\tau_3) \times \mathbf{1}_N + iL \times \Omega + J] \right) \ . \quad (29) \end{aligned}$$

The symbol trg (Trg) stands for the graded trace over matrices of dimension 8 ($8N$), respectively. The saddle–point condition yields for the saddle–point manifold σ_G the result

$$\sigma_G = -iQ, \quad Q = T^{-1}LT \ . \quad (30)$$

The matrices T obey Eq. (22) and belong to the coset space G/P defined by $G/P = \mathrm{U}(2,2/4)/\mathrm{U}(2/2) \times \mathrm{U}(2/2)$, where $P = \mathrm{U}(2/2) \times \mathrm{U}(2/2)$ is the subgroup of matrices in G which commute with L. After integration over the massive modes and taking $N \to \infty$, $\overline{Z(J)}$ takes the form

$$\overline{Z(J)} = \int \mathcal{D}[Q] \exp\left(i\mathcal{L}_{\mathrm{eff}}(Q) + i\mathcal{L}_{\mathrm{source}}(Q, J) \right) \ , \quad (31)$$

where

$$i\mathcal{L}_{\mathrm{eff}}(Q) = \frac{t}{2}\ \mathrm{trg}\ [(Q\tau_3)^2] - \mathrm{Trg}\ \ln\ [i\lambda(Q + \frac{t}{N}\tau_3 Q\tau_3) \times \mathbf{1}_N + iL \times \Omega] \quad (32)$$

and

$$\begin{aligned} & i\mathcal{L}_{\mathrm{source}}(Q, J) = \\ & -\mathrm{Trg}\ \ln \left[\mathbf{1}_8 \times \mathbf{1}_N + \left(i\lambda(Q + \frac{t}{N}\tau_3 Q\tau_3) \times \mathbf{1}_N + iL \times \Omega \right)^{-1} J \right] . (33) \end{aligned}$$

Summing over the level index μ, introducing the transmission coeffients T_c of Eq. (10) and recalling that $T_c = 1$ for all c, we obtain

$$\overline{g^{(1)}g^{(2)}} = \int \mathcal{D}[Q]\ \mathrm{detg}(1+QL)^{-M}\ R(Q)\ \exp\left(-\frac{t}{2}\ \mathrm{trg}[(Q\tau_3)^2]\right)\ , \qquad (34)$$

where the source term $R(Q)$ has the form

$$\begin{aligned} R(Q) = &\frac{1}{2}M^2\ \mathrm{trg}[G^{(11)}G^{(22)}]\mathrm{trg}[G^{(21)}G^{(12)}] \\ &+\frac{1}{4}M^4\ \mathrm{trg}[G^{(11)}G^{(12)}]\mathrm{trg}[G^{(21)}G^{(22)}] \\ &+\frac{1}{4}M^3\ \mathrm{trg}[G^{(11)}G^{(12)}G^{(21)}G^{(22)} + G^{(11)}G^{(22)}G^{(21)}G^{(12)}]\ , \end{aligned} \qquad (35)$$

with

$$G^{(rp)} = G^{(rp)}(Q) = (1+QL)^{-1}I^{(rp)}\ . \qquad (36)$$

The autocorrelation function is a function of two parameters, the channel number M and the parameter t for the magnetic field,

$$C(t, M) = \overline{\delta g^{(1)}\delta g^{(2)}} = \overline{g^{(1)}g^{(2)}} - \overline{g^{(1)}}\,\overline{g^{(2)}}\ , \qquad (37)$$

with $\overline{g^{(1)}g^{(2)}}$ given by the graded integral (34). The graded matrices Q can be parametrized in terms of 32 real variables, half of them commuting, the others, anticommuting. In calculations of the average two–point function, one typically deals with a total of 16 integration variables. Except for this increase in the number of variables and for the form of the source terms (which are, of course, specific to our problem), the form of our result in Eqs. (37) and (34) is quite standard. In spite of this similarity, the increase in the number of variables renders a full analytical evaluation of the graded integral (34) very difficult. As pointed out in the Introduction, our analytical work is restricted to the region of small t. We evaluate the leading terms of the asymptotic expansion of $C(t, M)$ in powers of t. Progress in this calculation depends crucially on the proper choice of the 32 variables used in parametrizing the matrices Q.

2 Parametrization of the Manifold

The integral (34) extends over the manifold of $Q = T^{-1}LT$, where T stands for the graded matrices of the coset space $\mathrm{U}(2,2/4)/\mathrm{U}(2/2)\times\mathrm{U}(2/2)$. Exponentiating the coset generators shows that the matrices T can be parametrized by two four-dimensional graded matrices t_{12} and t_{21},

$$T = \begin{pmatrix} \sqrt{1+t_{12}t_{21}} & t_{12} \\ t_{21} & \sqrt{1+t_{21}t_{12}} \end{pmatrix} , \tag{38}$$

related to each other by

$$t_{21} = kt_{12}^{\dagger} , \tag{39}$$

with $k = \mathrm{diag}(1,1,-1,-1)$. (In presenting the matrices explicitly, we assume that the row labels of matrices of dimensions 8, 4, and 2 are (p,α,r), (α,r), and by r, respectively, and that the indices follow in lexicographical order. The matrices of dimensions 8 and 4 are presented in block form.) The 32 matrix elements of t_{12} and t_{21} represent the Cartesian coordinates of T. The range of the commuting coordinates is defined by the condition that the ordinary part of the matrix $1 + t_{12}t_{21}$ be positive definite (non–compact parametrization in Boson–Boson blocks, compact parametrization in Fermion–Fermion blocks). Following the arguments of section 5.5 of VWZ, we find that the integration measure $\mathcal{D}[Q]$ is identical with the invariant integration measure for integration over the coset space. In Cartesian coordinates, this measure takes the form

$$\mathrm{d}\mu(T(t_{12},t_{21})) = \prod_{\alpha r,\alpha' r'} \mathrm{d}[t_{12}]_{\alpha r,\alpha' r'}\mathrm{d}[t_{12}]^{\star}_{\alpha r,\alpha' r'} , \tag{40}$$

where for ordinary complex $z = x + iy$, our $\mathrm{d}z\mathrm{d}z^{\star} = 2i\mathrm{d}x\mathrm{d}y$. We write $\mathrm{d}\mu(T)$ for $\mathrm{d}\mu(T(t_{12},t_{21}))$. Despite the simplicity of the measure, the Cartesian coordinates are not well suited for evaluating our integrals [11]. Therefore, we follow Efetov and change to polar coordinates.

2.1 Polar Coordinates

As in VWZ, we express the matrices $t_{12}t_{21}$ and $t_{21}t_{12}$ in terms of their common eigenvalue matrix ϵ and of the eigenvector matrices u_1 and u_2,

$$t_{12}t_{21} = u_1\epsilon u_1^{-1}, \quad t_{21}t_{12} = u_2\epsilon u_2^{-1} . \tag{41}$$

The unitarity relation (39) requires that both diagonalizing matrices u_1 and u_2 belong to graded unitary groups of type U(2/2),

$$u_1^{-1} = u_1^\dagger, \quad u_2^{-1} = k u_2^\dagger k \ . \tag{42}$$

To specify the matrices ϵ, u_1 and u_2 unambiguously, we order the eigenvalues $\epsilon_{\alpha 1}$ and $\epsilon_{\alpha 2}$ by the magnitudes of their ordinary parts, $\mathrm{ord}|\epsilon_{b1}| < \mathrm{ord}|\epsilon_{b2}|$, $\mathrm{ord}|\epsilon_{f1}| < \mathrm{ord}|\epsilon_{f2}|$. Moreover, we assume that the diagonalizing matrices u_1, u_2 are elements of the cosets $\mathrm{U}(2/2)/\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$, given by

$$u_1 = u_{1x} u_{1\beta} u_{1\gamma}, \quad u_2 = u_{2x} u_{2\beta} u_{2\gamma} \ . \tag{43}$$

Here

$$\begin{aligned}
u_{1x} &= \exp\begin{pmatrix} 0 & \xi_{1b} & 0 & 0 \\ -\xi_{1b}^\star & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_{1f} \\ 0 & 0 & -\xi_{1f}^\star & 0 \end{pmatrix}, \ u_{2x} = \exp\begin{pmatrix} 0 & \xi_{2b} & 0 & 0 \\ -\xi_{2b}^\star & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_{2f} \\ 0 & 0 & -\xi_{2f}^\star & 0 \end{pmatrix}, \\
u_{1\beta} &= \exp\begin{pmatrix} 0 & 0 & 0 & \beta_{11} \\ 0 & 0 & \beta_{12} & 0 \\ 0 & \beta_{12}^\star & 0 & 0 \\ \beta_{11}^\star & 0 & 0 & 0 \end{pmatrix}, \ u_{2\beta} = \exp\begin{pmatrix} 0 & 0 & 0 & i\beta_{21} \\ 0 & 0 & i\beta_{22} & 0 \\ 0 & i\beta_{22}^\star & 0 & 0 \\ i\beta_{21}^\star & 0 & 0 & 0 \end{pmatrix}, \\
u_{1\gamma} &= \exp\begin{pmatrix} 0 & 0 & \gamma_{11} & 0 \\ 0 & 0 & 0 & \gamma_{12} \\ \gamma_{11}^\star & 0 & 0 & 0 \\ 0 & \gamma_{12}^\star & 0 & 0 \end{pmatrix}, \ u_{2\gamma} = \exp\begin{pmatrix} 0 & 0 & i\gamma_{21} & 0 \\ 0 & 0 & 0 & i\gamma_{22} \\ i\gamma_{21}^\star & 0 & 0 & 0 \\ 0 & i\gamma_{22}^\star & 0 & 0 \end{pmatrix}
\end{aligned} \tag{44}$$

are exponentials of coset generators specified by 8 commuting parameters $\xi_{p\alpha}$, $\xi_{p\alpha}^\star$, and by 16 anticommuting parameters β_{pq}, $\beta_{pq}^\star$, γ_{pr} and $\gamma_{pr}^\star$, with $\alpha = b, f$, and $p, q, r = 1, 2$. We parametrize the matrices u_{px} by their off-diagonal matrix elements $x_{p\alpha} = \mathrm{e}^{i \arg \xi_{p\alpha}} \sin|\xi_{p\alpha}|$. We note that the matrices $u_1^{-1} t_{12} u_2$ and $u_2^{-1} t_{21} u_1$ commute with the eigenvalue matrix ϵ, denote these two diagonal matrices by λ and $\bar{\lambda}$, respectively, and write the matrices t_{12} and t_{21} as products,

$$t_{12} = u_1 \lambda u_2^{-1}, \quad t_{21} = u_2 \bar{\lambda} u_1^{-1} \ . \tag{45}$$

According to Eqs. (39) and (41), the matrices λ and $\bar{\lambda}$ are related to each other and to the eigenvalue matrix ϵ by

$$\bar{\lambda} = k\lambda^{\star}, \quad \epsilon = \lambda\bar{\lambda}. \tag{46}$$

Substituting Eq.(45) in Eq.(38) yields

$$T = U\Lambda U^{-1}, \quad U = U_x U_\beta U_\gamma,$$

$$U = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \sqrt{1+\lambda\bar{\lambda}} & \lambda \\ \bar{\lambda} & \sqrt{1+\bar{\lambda}\lambda} \end{pmatrix} , \tag{47}$$

with

$$U_w = \mathrm{diag}(u_{1w}, u_{2w}), \quad w = x, \beta, \gamma, \quad \lambda = \mathrm{diag}(\lambda_{b1}, \lambda_{b2}, \lambda_{f1}, \lambda_{f2}) \ . \tag{48}$$

These equations define the desired polar parametrization of T. Out of 32 polar coordinates, 24 reside in the matrix U (parameters $x_{p\alpha}, \beta_{pq}, \gamma_{pr}$, with $\alpha = b, f$ and $p, q, r = 1, 2$, and their complex conjugates), 8 in the matrix Λ (parameters $\lambda_{\alpha r}$, with $\alpha = b, f$ and $r = 1, 2$, and their complex conjugates).

For the assumed ordering of the eigenvalues $\epsilon_{\alpha r}$, the range of the absolute values of the ordinary parts of the commuting polar variables is specified by the inequalities

$$\begin{aligned} &0 < \mathrm{ord}|\lambda_{b1}| < \mathrm{ord}|\lambda_{b2}| < \infty \ , \\ &0 < \mathrm{ord}|\lambda_{f1}| < \mathrm{ord}|\lambda_{f2}| < 1 \ , \\ &0 < \mathrm{ord}|x_{p\alpha}| < 1 \ . \end{aligned} \tag{49}$$

The phase angles of the ordinary parts are allowed to take on all values between 0 and 2π. The integration measure $\mathrm{d}\mu(U\Lambda U^{-1})$ in polar coordinates is obtained from the Berezinian [15]. We get

$$\mathrm{d}\mu(U\Lambda U^{-1}) = \mathrm{d}\mu(U)\mathrm{d}\mu(\Lambda) \ . \tag{50}$$

Here $\mathrm{d}\mu(U)$ denotes the measure for integration over the matrices U, a product of measures for integration over the matrices $U_{x\beta} = U_x U_\beta$ and U_γ,

$$\mathrm{d}\mu(U) = \mathrm{d}\mu(U_{x\beta})\mathrm{d}\mu(U_\gamma) \ . \tag{51}$$

We have defined

$$\mathrm{d}\mu(U_{x\beta}) = \prod_{p\alpha} \mathrm{d}x_{p\alpha}\mathrm{d}x^\star_{p\alpha} \cdot \prod_{pq} \mathrm{d}\beta_{pq}\mathrm{d}\beta^\star_{pq} \cdot \prod_{p}(1 - 2\beta_{p1}\beta^\star_{p1}\beta_{p2}\beta^\star_{p2}) \ , \tag{52}$$

$$\mathrm{d}\mu(U_\gamma) = \prod_{pr} \mathrm{d}\gamma_{pr}\mathrm{d}\gamma^\star_{pr} \ . \tag{53}$$

The quantity $\mathrm{d}\mu(\Lambda)$ denotes the measure for integration over the matrices Λ,

$$\mathrm{d}\mu(\Lambda) = \prod_{\alpha r} \mathrm{d}\lambda_{\alpha r}\mathrm{d}\lambda^\star_{\alpha r} \cdot \prod_{\alpha}(|\lambda_{\alpha 1}|^2 - |\lambda_{\alpha 2}|^2)^2 \cdot \prod_{rr'}(|\lambda_{br}|^2 + |\lambda_{fr'}|^2)^{-2} \ . \tag{54}$$

Eq. (54) shows that the measure $\mathrm{d}\mu(U\Lambda U^{-1})$ contains a nonintegrable singularity. It is located at $\mathrm{ord}(|\lambda_{b1}|^2 + |\lambda_{f1}|^2) = 0$, i.e., at the surface of the domain of integration. This singularity causes the occurrence of an additional term, the Efetov–Wegner term.

2.2 The Efetov–Wegner Term

For pedagogical reasons, we first exemplify the treatment of the singularity for a simpler case. Indeed, the same problem arises in the integration over the four–dimensional matrices T_a belonging to the coset space $\mathrm{U}(1,1/2)/\mathrm{U}(1/1)\times \mathrm{U}(1/1)$. We return to the full problem below.

The properties of the four–dimensional matrices T_a are the same as those of their eight–dimensional counterparts T. The coset space consists of the matrices T_a satisfying $T_a^{-1} = s_a T_a^\dagger s_a$, now with $s_a = \mathrm{diag}(1,1,-1,1)$, modulo the subgroup of matrices which commute with $L_a = \mathrm{diag}(1,1,-1,-1)$. (We label the rows of graded matrices of dimensions 4 and 2 by the indices (p,α) and α, respectively.) The matrices have the form

$$T_a = \begin{pmatrix} \sqrt{1 + t^a_{12}t^a_{21}} & t^a_{12} \\ t^a_{21} & \sqrt{1 + t^a_{21}t^a_{12}} \end{pmatrix} \tag{55}$$

where the two-dimensional blocks t^a_{12} and t^a_{21} are related to each other by

$$t^a_{21} = k_a[t^a_{12}]^\dagger, \tag{56}$$

with $k_a = \mathrm{diag}(1,-1)$. The matrix elements of these blocks contain the 8 Cartesian coordinates of T_a. Again, the range of the commuting coordinates

is defined by the condition that the ordinary part of $1 + t_{12}t_{21}$ be positive definite. We consider the integral

$$\mathcal{I} = \int \mathrm{d}\mu(T_a) F(T_a) \tag{57}$$

of a function $F(T_a)$ over the coset space with the measure

$$\mathrm{d}\mu(T_a) = \prod_{\alpha\alpha'} \mathrm{d}[t^a_{12}]_{\alpha\alpha'} \mathrm{d}[t^a_{12}]^\star_{\alpha\alpha'} \ . \tag{58}$$

Polar coordinates are introduced [11] in the same way as described above. This yields $T_a = U_a \Lambda_a U_a^{-1}$. The anticommuting coordinates reside in the eigenvector matrix U_a, and the commuting ones, in the "eigenvalue matrix" Λ_a. We have

$$U_a = \begin{pmatrix} u_{1a} & 0 \\ 0 & u_{2a} \end{pmatrix} , \quad \Lambda_a = \begin{pmatrix} \sqrt{1 + \lambda_a \bar{\lambda}_a} & \lambda_a \\ \bar{\lambda}_a & \sqrt{1 + \bar{\lambda}_a \lambda_a} \end{pmatrix} , \tag{59}$$

where u_{pa} and λ_a denote the two–dimensional matrices

$$u_{1a} = \exp \begin{pmatrix} 0 & \gamma_{1a} \\ \gamma^\star_{1a} & 0 \end{pmatrix} , \quad u_{2a} = \exp \begin{pmatrix} 0 & i\gamma_{2a} \\ i\gamma^\star_{2a} & 0 \end{pmatrix} , \quad \lambda_a = \begin{pmatrix} \lambda_{ba} & 0 \\ 0 & \lambda_{fa} \end{pmatrix} , \tag{60}$$

and where $\bar{\lambda}_a = k_a \lambda^\star_a$. Moreover,

$$t^a_{12} = u_{1a} \lambda_a u^{-1}_{2a} , \quad t^a_{21} = u_{2a} \bar{\lambda}_a u^{-1}_{1a} \ . \tag{61}$$

For the range of the ordinary parts of the commuting polar variables, we have $0 < \mathrm{ord}|\lambda_{ba}| < \infty$, $0 < \mathrm{ord}|\lambda_{fa}| < 1$. The integration measure in polar coordinates is given by

$$\begin{aligned} \mathrm{d}\mu(U_a \Lambda_a U_a^{-1}) &= \mathrm{d}\mu(U_a) \mathrm{d}\mu(\Lambda_a) \ , \\ \mathrm{d}\mu(U_a) &= \prod_p \mathrm{d}\gamma_{pa} \mathrm{d}\gamma^\star_{pa} \ , \\ \mathrm{d}\mu(\Lambda_a) &= \prod_\alpha \mathrm{d}\lambda_{\alpha a} \mathrm{d}\lambda^\star_{\alpha a} \cdot (|\lambda_{ba}|^2 + |\lambda_{fa}|^2)^{-2} \ . \end{aligned} \tag{62}$$

The last line in Eq. (62) shows that the measure again contains a nonintegrable singularity located at $\mathrm{ord}(|\lambda_{ba}|^2 + |\lambda_{fa}|^2) = 0$. Thus, the integral

cannot be done by straightforwardly substituting $T_a = U_a \Lambda_a U_a^{-1}$ in F and integrating with the measure (62). This can be seen by the following example. Suppose that F depends only on the eigenvalues $\lambda_{ba}, \lambda_{fa}$, and does not vanish at $|\lambda_{ba}|^2 + |\lambda_{fa}|^2{=}0$. Despite the fact that the integral over F in the Cartesian coordinates is well defined, the straightforward integration in polar coordinates gives an indefinite expression of the type $0 \cdot \infty$, with 0 and ∞ resulting from the integration over the anticommuting and commuting polar variables, respectively. To derive the correct formula for the integration in polar coordinates, we proceed in the following way. We start with the Cartesian coordinates, exclude from the domain of integration an infinitesimal neighbourhood of the singularity of $\mathrm{d}\mu(U_a \Lambda_a U_a^{-1})$, and express the integral $\mathcal{I}$ as the limit

$$\mathcal{I} = \lim_{\varepsilon \to 0} \int \mathrm{d}\mu(T_a) F(T_a) \theta(v(T_a) - \varepsilon) \ . \tag{63}$$

Here ε is real and positive, and $v(T_a)$ represents the ordinary part of $|\lambda_{ba}|^2 + |\lambda_{fa}|^2$, as given in terms of Cartesian coordinates by

$$v(T_a) = [t_{12}^a]_{bb} [t_{21}^a]_{bb} - [t_{12}^a]_{ff} [t_{21}^a]_{ff} \ . \tag{64}$$

Since the integral on the right-hand-side of Eq. (63) is over the region where the measure $\mathrm{d}\mu(U_a \Lambda_a U_a^{-1})$ is regular, this integral can be evaluated by the straightforward change to polar coordinates. Thus we have

$$\mathcal{I} = \lim_{\varepsilon \to 0} \int \mathrm{d}\mu(U_a \Lambda_a U_a^{-1}) F(U_a \Lambda_a U_a^{-1}) \theta(v(U_a \Lambda_a U_a^{-1}) - \varepsilon) \ . \tag{65}$$

We write the function $v = v_c + v_n$ as the sum of the part $v_c = |\lambda_{ba}|^2 + |\lambda_{fa}|^2$ which contains only the commuting variables λ_{ba} and λ_{fa}, and the nilpotent complement v_n,

$$\begin{aligned} v_n = {} & (|\lambda_{ba}|^2 - |\lambda_{fa}|^2)(\gamma_{1a}\gamma_{1a}^\star - \gamma_{2a}\gamma_{2a}^\star) \\ & + 2i(\lambda_{ba}\lambda_{fa}^\star \gamma_{2a}\gamma_{1a}^\star - \lambda_{fa}\lambda_{ba}^\star \gamma_{1a}\gamma_{2a}^\star) \\ & - 2\ (|\lambda_{ba}|^2 + |\lambda_{fa}|^2)\gamma_{1a}\gamma_{1a}^\star \gamma_{2a}\gamma_{2a}^\star \ . \end{aligned} \tag{66}$$

We expand $\theta(v_c + v_n - \varepsilon)$ in a Taylor series at $v_c - \varepsilon$. The integral (65) decomposes into two terms,

$$\mathcal{I} = \mathcal{I}_V + \mathcal{I}_W \ . \tag{67}$$

The volume term has the form

$$\mathcal{I}_V = \lim_{\varepsilon\to 0}\int d\mu(U_a\Lambda_a U_a^{-1})F(U_a\Lambda_a U_a^{-1})\theta(v_c-\varepsilon)\ , \tag{68}$$

and the boundary term or Efetov–Wegner term is given by

$$\mathcal{I}_W = \lim_{\varepsilon\to 0}\int d\mu(U_a\Lambda_a U_a^{-1})F(U_a\Lambda_a U_a^{-1})(\delta(v_c-\varepsilon)v_n + \frac{1}{2}\delta'(v_c-\varepsilon)v_n^2)\ . \tag{69}$$

Terms of higher order in v_n vanish. To calculate $\mathcal{I}_W$, we introduce new commuting coordinates ρ, ϕ, ϕ_b and ϕ_f by

$$\lambda_{ba} = \sqrt{\rho}\cos\phi\, e^{i\phi_b}, \quad \lambda_{fa} = \sqrt{\rho}\sin\phi\, e^{i\phi_f}. \tag{70}$$

Since v_n is linear in ρ, the singularity ρ^{-1} in the measure is cancelled by the factors ρ and ρ^2 arising in the terms due to the Taylor expansion in Eq. (69). Because of the terms $\delta(\rho-\varepsilon)$ and $\rho\delta'(\rho-\varepsilon)$ in the integrand, the function $F(U_a\Lambda_a U_a^{-1})$ can be set equal to its value at $T=1$. With

$$v_n^2 = -2\rho^2\gamma_{1a}\gamma_{1a}^\star\gamma_{2a}\gamma_{2a}^\star, \qquad \int d\mu(U_a)v_n^j = -\frac{2}{(2\pi)^2}\rho^j, \quad j=1,2, \tag{71}$$

we find that

$$\mathcal{I}_W = F(1)\,\frac{1}{(2\pi)^2}\lim_{\varepsilon\to 0}\int d\rho d\phi\, d\phi_b d\phi_f \sin 2\phi\ \delta(\rho-\varepsilon) = F(1)\ . \tag{72}$$

In summary, we have

$$\mathcal{I} = \lim_{\varepsilon\to 0}\int d\mu(U_a\Lambda_a U_a^{-1})F(U_a\Lambda_a U_a^{-1})\theta(|\lambda_{ba}|^2+|\lambda_{fa}|^2-\varepsilon) + F(1)\ . \tag{73}$$

This is the desired result. It expresses the integral as a volume term and a surface contribution.

The limit in the volume term can be taken after using the Taylor expansion of the function $F(U_a\Lambda_a U_a^{-1})$ in the Grassmann variables γ_{pa},$\gamma_{pa}^\star$. Except for the term of zeroth order, $F_0(U_a\Lambda_a U_a^{-1})$, all higher terms in the expansion tend to zero as Λ_a tends to the unit matrix. We assume that these terms vanish as $(|\lambda_{ba}|^2+|\lambda_{fa}|^2)^\alpha$ with $\alpha\geq 1$. Then, only the expansion term of fourth order $F_4(U_a\Lambda_a U_a^{-1})$ contributes. Working out the limit yields

$$\mathcal{I} = \mathcal{I}_V + \mathcal{I}_W = \int d\mu(U_a\Lambda_a U_a^{-1})F_4(U_a\Lambda_a U_a^{-1}) + F_0(1)\ . \tag{74}$$

We apply this method to an integral over a function $F(T)$ depending on the eight–dimensional matrix T,

$$\mathcal{I} = \int \mathrm{d}\mu(T) F(T). \tag{75}$$

From Subsection 2.1 it follows that the matrix T is given by the product

$$T = U_{x\beta}\ \mathrm{diag}(T_1, T_2)\ U_{x\beta}^{-1}, \tag{76}$$

where

$$T_r = \begin{pmatrix} \sqrt{1 + t_{12}^r t_{21}^r} & t_{12}^r \\ t_{21}^r & \sqrt{1 + t_{21}^r t_{12}^r} \end{pmatrix}, \quad r = 1, 2, \tag{77}$$

are the elements of the coset space $\mathrm{U}(1, 1/2)/\mathrm{U}(1/1) \times \mathrm{U}(1/1)$ with polar coordinates γ_{pr},$\gamma_{pr}^\star$ and $\lambda_{\alpha r}$,$\lambda_{\alpha r}^\star$ (cf. Eqs. (59) and (60)),

$$T_r = U_r \Lambda_r U_r^{-1}, \quad U_r = \begin{pmatrix} u_{1r} & 0 \\ 0 & u_{2r} \end{pmatrix}, \quad \Lambda_r = \begin{pmatrix} \sqrt{1 + \lambda_r \bar{\lambda}_r} & \lambda_r \\ \bar{\lambda}_r & \sqrt{1 + \bar{\lambda}_r \lambda_r} \end{pmatrix},$$

$$u_{1r} = \exp \begin{pmatrix} 0 & \gamma_{1r} \\ \gamma_{1r}^\star & 0 \end{pmatrix}, \quad u_{2r} = \exp \begin{pmatrix} 0 & i\gamma_{2r} \\ i\gamma_{2r}^\star & 0 \end{pmatrix}, \quad \lambda_r = \begin{pmatrix} \lambda_{br} & 0 \\ 0 & \lambda_{fr} \end{pmatrix} . \tag{78}$$

In Eq. (76), we now regard the matrices T_r as functions of t_{12}^r and t_{21}^r. This defines a new parametrization of T in terms of the polar coordinates of $U_{x\beta}$ and the Cartesian coordinates of T_1 and T_2, with integration measure

$$\mathrm{d}(U_{x\beta}\ \mathrm{diag}(T_1, T_2)\ U_{x\beta}^{-1}) = \mathrm{d}\mu(U_{x\beta}) \mathrm{d}\mu(T_1) \mathrm{d}\mu(T_2) m(\Lambda). \tag{79}$$

Here $\mathrm{d}\mu(U_{x\beta})$ denotes the measure (52) for integration over the matrices $U_{x\beta}$, $\mathrm{d}\mu(T_r)$ denotes the Cartesian measures for integration over the matrices T_r (cf. Eq. (58)),

$$\mathrm{d}\mu(T_r) = \prod_{\alpha\alpha'} \mathrm{d}[t_{12}^r]_{\alpha\alpha'} \mathrm{d}[t_{12}^r]_{\alpha\alpha'}^\star\ , \tag{80}$$

and $m(\Lambda)$ denotes the factor

$$m(\Lambda) = \prod_\alpha (|\lambda_{\alpha 1}|^2 - |\lambda_{\alpha 2}|^2)^2 \cdot \prod_{r \neq r'} (|\lambda_{br}|^2 + |\lambda_{fr'}|^2)^{-2}\ . \tag{81}$$

This measure does not contain any nonintegrable singularity in the domain of integration. Hence, we can write

$$\mathcal{I} = \int \mathrm{d}\mu(U_{x\beta}\ \mathrm{diag}(T_1, T_2)\ U_{x\beta}^{-1}) F(U_{x\beta}\ \mathrm{diag}(T_1, T_2)\ U_{x\beta}^{-1})\ . \tag{82}$$

The source term (35) vanishes at $T = 1$ where $Q = L$. Thus in the case of Eqs. (34) and (89), the function $F(U_{x\beta}\ \mathrm{diag}(T_1, T_2)\ U_{x\beta}^{-1})$ tends to zero, $F(1) = 0$, when T_1 and T_2 tend to the unit matrix. In the integrals over T_1 and T_2, we change to polar coordinates, using the rule (74). This yields

$$\mathcal{I} = \mathcal{I}_V + \mathcal{I}_W \ , \tag{83}$$

$$\mathcal{I}_V = \int \mathrm{d}\mu(U\Lambda U^{-1}) F_{44}(U\Lambda U^{-1}),$$

$$\mathcal{I}_W = \int \mathrm{d}\mu(U_{x\beta}) \mathrm{d}\mu(U_2\Lambda_2 U_2^{-1}) F_{04}(U_{x\beta}\ \mathrm{diag}(1, U_2\Lambda_2 U_2^{-1})\ U_{x\beta}^{-1}) \ . \tag{84}$$

Here $F_{n_1 n_2}(U\Lambda U^{-1})$ denotes the part of $F(U\Lambda U^{-1})$ which is of order n_1 in $\gamma_{p1}, \gamma_{p1}^\star$ and of order n_2 in $\gamma_{p2}, \gamma_{p2}^\star$, and $\mathrm{d}\mu(U_2\Lambda_2 U_2^{-1})$ denotes the integration measure in polar coordinates for the matrices T_2,

$$\mathrm{d}\mu(U_2\Lambda_2 U_2^{-1}) = \mathrm{d}\mu(U_2)\mathrm{d}\mu(\Lambda_2),$$

$$\mathrm{d}\mu(U_2) = \prod_p \mathrm{d}\gamma_{p2}\mathrm{d}\gamma_{p2}^\star,$$

$$\mathrm{d}\mu(\Lambda_2) = \prod_\alpha \mathrm{d}\lambda_{\alpha 2}\mathrm{d}\lambda_{\alpha 2}^\star \cdot (|\lambda_{b2}|^2 + |\lambda_{f2}|^2)^{-2} \ . \tag{85}$$

This is the final result.

We simplify the notation by writing the integrals $\mathcal{I}_V$ and $\mathcal{I}_W$ as

$$\mathcal{I}_\Omega = \int_\Omega \mathrm{d}\mu(U\Lambda U^{-1}) F(U\Lambda U^{-1}) \ , \qquad \Omega = V,\, W \ . \tag{86}$$

For $\Omega = W$, the matrices U, Λ and the measure $\mathrm{d}\mu(U\Lambda U^{-1})$ are by definition given by

$$U = U_{x\beta}\,\mathrm{diag}(1, U_2) \ , \quad \Lambda = \mathrm{diag}(1, \Lambda_2) \ ,$$

$$\mathrm{d}\mu(U\Lambda U^{-1}) = \mathrm{d}\mu(U_{x\beta})\,\mathrm{d}\mu(U_2\Lambda_2 U_2^{-1}) \ . \tag{87}$$

It is always understood that the only non–vanishing contribution arises from that part of F which is of the highest possible order in $\gamma_{pr}, \gamma_{pr}^\star$, i.e., of 8th order for $\Omega = V$, and of 4th order for $\Omega = W$.

3 Asymptotic Expansion for Small t

The behaviour of the correlator $C(t,M) = \overline{\delta g^{(1)} \delta g^{(2)}}$ at small t is described by the leading terms of the asymptotic expansion

$$C(t,M) = \overline{g^{(1)} g^{(2)}} - \overline{g^{(1)}}\;\overline{g^{(2)}} = \sum_{n=0}^{\infty} c(n,M)\, t^n - (M/2)^2 \ , \tag{88}$$

generated by expanding the exponential in the integrand of Eq. (34) in powers of t. The expansion coefficients $c(n,M)$ are given by the graded integrals

$$c(n,M) = \mathcal{N}_n \int \mathrm{d}\mu(T)\, D(M,Q)\, R(Q)\, S^n(Q) \ , \quad Q = T^{-1} L T \ , \tag{89}$$

where

$$\mathcal{N}_n = (-)^n/(2^n n!) \ , \quad D(M,Q) = \mathrm{detg}^{-M}(1 + QL) \ ,$$

$$S(Q) = \mathrm{trg}\,[(Q\tau_3)^2] \ , \tag{90}$$

and where $R(Q)$ denotes the source term (35). We note that by definition, the channel number M assumes even values only. For given M, only the first $M/2{+}1$ terms of the expansion (88) are meaningful: for higher values of n, the factors $S^n(Q)$ cause the integrals (89) to diverge (diverging integrals over the bosonic eigenvalues). We calculate the first three terms of the expansion. The expansion coefficients (89) can be worked out analytically. The calculation relies on symbolic manipulation utilities (we have used those of Mathematica) in an essential way and is described in Subsection 3.1. The results are given in Subsection 3.2.

3.1 The Coefficients $c(n,M)$ for $n \leq 2$

We change to polar coordinates, use the integration formulae (83) and (86), and write

$$c(n,M) = \sum_{\Omega} c_{\Omega}(n,M) \ , \tag{91}$$

where

$$c_{\Omega}(n,M) = \mathcal{N}_n \int_{\Omega} \mathrm{d}\mu(U\Lambda U^{-1})\, D(M,Q)\, R(Q)\, S^n(Q) \ ,$$

$$Q = U\Lambda^{-1} L \Lambda U^{-1} \ . \tag{92}$$

Both the volume part $c_V(n,M)$ and the boundary part $c_W(n,M)$ are evaluated in the same way. We only sketch the main steps.

Following VWZ, we express the complex variables $\lambda_{\alpha r}$ in terms of absolute values and phase angles,

$$\lambda_{\alpha r} = i \sin \frac{1}{2}\theta_{\alpha r}\, \mathrm{e}^{i\phi_{\alpha r}} \ , \qquad \theta_{br} = i\vartheta_{br} \ , \qquad \theta_{fr} = \vartheta_{fr} \ , \tag{93}$$

with $\vartheta_{\alpha r}$ and $\phi_{\alpha r}$ real. Substituting this expression in $T = U\Lambda U^{-1}$ and $Q = U\Lambda^{-1}L\Lambda U^{-1}$ yields

$$T = \tilde{U}\tilde{\Lambda}\tilde{U}^{-1} \ , \qquad \tilde{\Lambda} = \begin{pmatrix} \cos\frac{1}{2}\theta & i\sin\frac{1}{2}\theta \\ i\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta \end{pmatrix} \ , \tag{94}$$

$$\tilde{Q} = \tilde{U}\tilde{\Lambda}^{-1}L\tilde{\Lambda}\tilde{U}^{-1} \ , \qquad \tilde{\Lambda}^{-1}L\tilde{\Lambda} = \begin{pmatrix} \cos\theta & i\sin\theta \\ -i\sin\theta & -\cos\theta \end{pmatrix} \ , \tag{95}$$

where

$$\tilde{U} = \mathrm{diag}(\tilde{u}_1, \tilde{u}_2) \ , \qquad \tilde{u}_1 = u_1 \mathrm{e}^{i\phi} \ , \qquad \tilde{u}_2 = u_2 \ . \tag{96}$$

The matrices $\tilde{U}, \tilde{\Lambda}$ differ from their counterparts U, Λ only in their dependence on the phases $\phi_{\alpha r}$. We express the matrices T and Q in terms of the modified matrices $\tilde{U} = \mathrm{diag}(\tilde{u}_1, \tilde{u}_2)$ and $\tilde{\Lambda}$, and omit the tilde. It is advantageous to parametrize the modified "eigenvalue" matrix Λ in terms of

$$\mu_{\alpha r} = \cos^2 \frac{1}{2}\theta_{\alpha r} \ . \tag{97}$$

Explicit expressions for the modified matrices U and Λ and for the associated integration measures are provided in Appendix 6.1.

Inverting the matrix $1 + \Lambda^{-1}L\Lambda L$, we obtain

$$(1+QL)^{-1} = U(1+\Lambda^{-1}L\Lambda L)^{-1}U^{-1} = U\frac{1}{2}\begin{pmatrix} 1 & i\tan\frac{1}{2}\theta \\ i\tan\frac{1}{2}\theta & 1 \end{pmatrix}U^{-1} \ . \tag{98}$$

The term $D(M,Q)$ depends only on the matrix Λ,

$$D(M,Q) = \mathrm{detg}^{-M}(1+\Lambda^{-1}L\Lambda L) = D(M, \Lambda^{-1}L\Lambda) \ , \tag{99}$$

with

$$D(M,\Lambda^{-1}L\Lambda) \;=\; \begin{cases} (\mu_{f1}\mu_{f2})^M(\mu_{b1}\mu_{b2})^{-M} & \text{for} \quad \Omega = V \;, \\ \mu_{f2}^M\mu_{b2}^{-M} & \phantom{\text{for}} \quad \Omega = W \;. \end{cases} \tag{100}$$

Substituting Eq. (98) into the matrices $G^{(rp)}(Q) = (1+QL)^{-1}I^{(rp)}$ of the source term $R(Q)$ and separating the parts which contain different powers of M, we get

$$R(Q) = R_2(Q) + R_3(Q) + R_4(Q) \;, \tag{101}$$

where $R_k(Q)$, $k = 2, 3, 4$, denote the source term components

$$\begin{aligned} R_2(Q) &= \frac{M^2}{32}\,\mathrm{trg}\Big[\,\chi I^{(1)}(u_1)\chi I^{(2)}(u_2)\,\Big]\mathrm{trg}\Big[\,\chi I^{(2)}(u_1)\chi I^{(1)}(u_2)\,\Big] \;, \\ R_3(Q) &= \frac{M^3}{64}\,\mathrm{trg}\Big[\,\chi I^{(1)}(u_1)\chi I^{(1)}(u_2)\chi I^{(2)}(u_1)\chi I^{(2)}(u_2) \\ &\qquad +\chi I^{(1)}(u_1)\chi I^{(2)}(u_2)\chi I^{(2)}(u_1)\chi I^{(1)}(u_2)\,\Big] \;, \\ R_4(Q) &= \frac{M^4}{64}\,\mathrm{trg}\Big[\,\chi I^{(1)}(u_1)\chi I^{(1)}(u_2)\,\Big]\mathrm{trg}\Big[\,\chi I^{(2)}(u_1)\chi I^{(2)}(u_2)\,\Big] \;. \end{aligned} \tag{102}$$

The dependence of $R_k(Q)$ on U is contained in the graded matrices $I^{(r)}(u_p)$,

$$I^{(r)}(u_p) = u_p^{-1}I^{(r)}u_p \;, I^{(1)} = \mathrm{diag}(1,0,-1,0) \;, I^{(2)} = \mathrm{diag}(0,1,0,-1) \;, \tag{103}$$

their dependence on Λ in the matrix

$$\chi = \tan\frac{1}{2}\theta \;. \tag{104}$$

Substituting Eq. (95) in the coupling term $S(Q)$ yields

$$S(Q) = S_{11}(Q) + S_{22}(Q) + 2S_{12}(Q) \;, \tag{105}$$

where $S_{pp'}(Q)$ are the graded traces

$$
\begin{aligned}
S_{11}(Q) &= \operatorname{trg}\Big[\cos\theta\,\tau_3(u_1)\ \cos\theta\,\tau_3(u_1)\Big]\ ,\\
S_{22}(Q) &= \operatorname{trg}\Big[\cos\theta\,\tau_3(u_2)\ \cos\theta\,\tau_3(u_2)\Big]\ ,\\
S_{12}(Q) &= \operatorname{trg}\Big[\sin\theta\,\tau_3(u_1)\ \sin\theta\,\tau_3(u_2)\Big]\ ,
\end{aligned}
\tag{106}
$$

which depend on U via the graded matrices

$$
\tau_3(u_p) = u_p^{-1}\tau_3 u_p\ , \quad \tau_3 = \mathrm{diag}(1,-1,1,-1)\ . \tag{107}
$$

Writing the measure $\mathrm{d}\mu(U\Lambda U^{-1})$ as the product of measures for integration over U and over Λ, using Eq. (101) and performing first the integration over U gives the coefficients $c_\Omega(n,M)$ as the sum of contributions of the three source components,

$$
c_\Omega(n,M) = \sum_k c_\Omega^{(k)}(n,M)\ , \tag{108}
$$

where

$$
c_\Omega^{(k)}(n,M) = \int_\Omega \mathrm{d}\mu(\Lambda) D(M,\Lambda^{-1}L\Lambda) F_\Omega^{(k)}(n,M,\Lambda)\ , \tag{109}
$$

with

$$
F_\Omega^{(k)}(n,M,\Lambda) = \mathcal{N}_n \int_\Omega \mathrm{d}\mu(U) R_k(Q) S^n(Q)\ . \tag{110}
$$

The measure $\mathrm{d}\mu(U)$ is the product of measures for integration over the matrices u_1 and u_2, the source components $R_k(Q)$ and the traces $S_{pp'}(Q)$ depend on u_p only in terms of the matrices $I^{(r)}(u_p)$ and $\tau_3(u_p)$, and this dependence has the simple form shown of Eqs. (102) and (106). For all these reasons, the evaluation of $F_\Omega^{(k)}(n,M,\Lambda)$ reduces to the calculation of the integrals

$$
\begin{aligned}
&K_\Omega^{(p)}(i_1,j_1,i_2,j_2|k_1,l_1,\ldots,k_m,l_m)\\
&= \int_\Omega \mathrm{d}\mu(u_p)[I^{(1)}(u_p)]_{i_1j_1}[I^{(2)}(u_p)]_{i_2j_2}[\tau_3(u_p)]_{k_1l_1}\ldots[\tau_3(u_p)]_{k_ml_m}\ ,
\end{aligned}
\tag{111}
$$

where $p=1,2$. From the mathematical point of view, these integrals are linear forms of integrals of products of matrix elements of the coset matrices u_p, u_p^{-1} over the coset $U(2/2)/U(1)\times U(1)\times U(1)\times U(1)$, and could therefore be evaluated by a generating function approach similar to that worked out for the same integrals over the graded unitary group by Guhr [18, 19].

The integrals $K_\Omega^{(1)}$ and $K_\Omega^{(2)}$ referring to the same set of indices $i_1,j_1,i_2,j_2,k_1,l_1,\ldots,k_m,l_m$ are closely related to each other. Starting from the integral

$K_V^{(1)}(i_1,j_1,i_2,j_2|k_1,l_1,\ldots,k_m,l_m)$, making the change to the primed anticommuting variables introduced by $\beta_{1q}=i\beta'_{1q}$, $\beta^\star_{1q}=i\tilde\beta'_{1q}$, $\gamma_{1r}=i\gamma'_{1r}$, $\gamma^\star_{1r}=i\tilde\gamma'_{1r}$, and comparing the result with the integral $K_V^{(2)}(i_1,j_1,i_2,j_2|k_1,l_1,\ldots,k_m,l_m)$, we find that the two integrals in fact differ only by a factor, the integral over the phases $\phi_{\alpha r}$ which is present in $K_V^{(1)}$ and absent in $K_V^{(2)}$. Apart from an additional difference in sign due to the Berezinian, the same is true for the integrals $K_W^{(1)}$ and $K_W^{(2)}$. This simplifies the calculation in an essential way. The integration over the anticommuting variables yields the part of highest possible order of

$$(1\,-2\,\beta_{p1}\beta^\star_{p1}\beta_{p2}\beta^\star_{p2})[I^{(1)}(u_p)]_{i_1j_1}[I^{(2)}(u_p)]_{i_2j_2}[\tau_3(u_p)]_{k_1l_1}\ldots[\tau_3(u_p)]_{k_ml_m} \quad (112)$$

(of 16th order for $\Omega=V$, of 12th order for $\Omega=W$). After setting $x_{p\alpha}=\sin\zeta_{p\alpha}\mathrm{e}^{i\eta_{p\alpha}}$, the integration over the commuting variables appearing in u_p simplifies to integrals over phase angles, or over products of powers of basic trigonometric functions. Multiplying the products of complementary pairs of integrals $K_\Omega^{(p)}(i_1,j_1,i_2,j_2|k_1,l_1,\ldots,k_m,l_m)$ by the Λ-dependent factors stemming from χ and $\cos\theta$ and collecting the contributions yields the desired $F_\Omega^{(k)}(n,M,\Lambda)$. The result has the form

$$F_V^{(k)}(n,M,\Lambda)=\mathcal{N}_{nk}M^k\prod_{rr'}(\mu_{br}-\mu_{fr'})\cdot\prod_{\alpha r}\mu_{\alpha r}^{-2}\cdot\;P_V^{(k)}(n,\Lambda)\;, \quad (113)$$

$$F_W^{(k)}(n,M,\Lambda)=\mathcal{N}_{nk}M^k(\mu_{b2}-\mu_{f2})\prod_{\alpha}\mu_{\alpha 2}^{-2}\cdot P_W^{(k)}(n,\Lambda)\;, \quad (114)$$

where $P_\Omega^{(k)}(n,\Lambda)$ are polynomials in $\mu_{\alpha r}$, and where

$$\mathcal{N}_{nk}=(-)^n(1+\delta_{k2})/(2^{n+2}n!). \quad (115)$$

The polynomials $P_V^{(k)}(n,\Lambda)$ are symmetrical with respect to the interchange of μ_{b1} and μ_{b2}, and of μ_{f1} and μ_{f2}. Taking into account this property, we can write the final integrals over Λ as

$$\begin{aligned}
&c_V^{(k)}(n,M)\\
&=\mathcal{N}_{nk}M^k\int_V \mathrm{d}\mu(\Lambda)\prod_r\mu_{fr}^{M-2}\mu_{br}^{-M-2}\cdot\prod_{rr'}(\mu_{br}-\mu_{fr'})\cdot\;P_V^{(k)}(n,\Lambda)\\
&=\mathcal{N}_{nk}M^k\,\frac{1}{4}\int_1^\infty\mathrm{d}\mu_{b1}\int_1^\infty\mathrm{d}\mu_{b2}\int_1^0\mathrm{d}\mu_{f1}\int_1^0\mathrm{d}\mu_{f2}\;\times
\end{aligned}$$

$$\frac{\prod_r \mu_{fr}^{M-2}}{\prod_r \mu_{br}^{M+2}} \; \frac{\prod_\alpha \; (\mu_{\alpha 1} - \mu_{\alpha 2})^2}{\prod_{rr'} (\mu_{br} - \mu_{fr'})} \; P_V^{(k)}(n, \Lambda) \; ,$$

$$\begin{aligned} c_W^{(k)}(n, M) &= \mathcal{N}_{nk} M^k \int_W \mathrm{d}\mu(\Lambda) \mu_{f2}^{M-2} \mu_{b2}^{-M-2} (\mu_{b2} - \mu_{f2}) \, P_W^{(k)}(n, \Lambda) \\ &= \mathcal{N}_{nk} M^k \int_1^\infty \mathrm{d}\mu_{b2} \int_1^0 \mathrm{d}\mu_{f2} \frac{\mu_{f2}^{M-2}}{\mu_{b2}^{M+2}} \frac{1}{\mu_{b2} - \mu_{f2}} P_W^{(k)}(n, \Lambda) \; . \end{aligned} \tag{116}$$

The integrals can be done analytically using the approach described in Appendix 6.2. Collecting the contributions of all three source components finally yields the coefficients $c_\Omega(n, M)$.

The calculation of the volume integrals $c_V^{(k)}(n, M)$ can be simplified by using modified polar coordinates where

$$\begin{aligned} &T = U \Lambda U^{-1} \; , \quad U = \mathrm{diag}(u_1, u_2) \; , \\ &u_1 = u_{1\gamma} u_{1\beta} u_{1x} u_\phi \; , \quad u_2 = u_{2\gamma} u_{2\beta} u_{2x} \; , \end{aligned} \tag{117}$$

with $u_{p\gamma}$,$u_{p\beta}$,u_{px} and u_ϕ having the same form as in the old parametrization. The integration measures for both parametrizations coincide. Since the matrices $u_{p\gamma}$ commute with the matrix τ_3, the matrices $\tau_3(u_p)$ are now independent of $\gamma_{pr}, \gamma_{pr}^\star$. In the integrals $K_V^{(p)}$, the only part of the integrand which then depends on these anticommuting variables is the product of matrix elements of the matrices $I^{(r)}(u_p)$, and only the part of highest order in $\gamma_{pr}, \gamma_{pr}^\star$ contributes. Straightforward calculation shows that this part is obtained by replacing the matrices $u_{p\gamma}^{-1} I^{(r)} u_{p\gamma}$ in $I^{(r)}(u_p)$ by the matrices $(-)^p 2 \gamma_{pr} \gamma_{pr}^\star J^{(r)}$, where the $J^{(r)}$'s denote the projectors $J^{(1)} = \mathrm{diag}(1, 0, 1, 0)$ and $J^{(2)} = \mathrm{diag}(0, 1, 0, 1)$. This substitution simplifies the calculation very much. Unfortunately, no similar coordinate transformation has been found to simplify the boundary integrals.

3.2 Leading Terms

We begin with the expansion term of zeroth order $c(0, M)$. The integrals $K_\Omega^{(p)}$ which enter the calculation do not contain any of the matrix elements of $\tau_3(u_p)$. Integrating over u_p we find that $K_V^{(p)}(i_1, j_1, i_2, j_2|) = 0$ for all i_1, j_1, i_2, j_2. Thus $c_V^{(k)}(0, M) = 0$ for any k, and the coefficient $c(0, M)$ is

entirely determined by the boundary terms $c_W^{(k)}(0,M)$. We obtain, after combining the contributions from the integrals $K_W^{(p)}(i_1,j_1,i_2,j_2|)$, that

$$\begin{aligned} P_W^{(2)}(0,\Lambda) &= -\mu_{b2}+\mu_{f2} \ , \\ P_W^{(3)}(0,\Lambda) &= 2(\mu_{b2}+\mu_{f2}-2\mu_{b2}\mu_{f2}) \ , \\ P_W^{(4)}(0,\Lambda) &= P_W^{(2)}(0,\Lambda) \ . \end{aligned} \tag{118}$$

The integration over Λ then yields

$$c_W^{(2)}(0,M) = \frac{M^2}{2(M^2-1)} = -\, c_W^{(3)}(0,M) \ , \qquad c_W^{(4)}(0,M) = \frac{M^4}{4(M^2-1)} \cdot \tag{119}$$

The contributions $c_W^{(2)}(0,M)$ and $c_W^{(3)}(0,M)$ cancel each other, and the coefficient $c(0,M)$ is given by the boundary contribution of the source component $R_4(Q)$,

$$c(0,M) = c_W^{(4)}(0,M) - \frac{1}{4}M^2 = \frac{M^2}{4(M^2-1)} \cdot \tag{120}$$

By definition, $c(0,M) = C(0,M)$ gives the variance $\mathrm{Var}[g(B)]$ of the dimensionless conductance $g(B)$. The value (120) of $\mathrm{Var}[g(B)]$ agrees with the one derived by Baranger and Mello using a random S–matrix approach [20].

We turn to the coefficient of the term linear in t. Here, we meet a similar situation: All integrals $K_V^{(p)}(i_1,j_1,i_2,j_2|k_1,l_1)$ vanish. With

$$\begin{aligned} P_W^{(2)}(1,\Lambda) &= (8/3) \ \times \\ & \quad \{\mu_{b2}(2-\mu_{b2}-3\mu_{b2}^2) - \mu_{b2}^2(2\mu_{b2}-3)\mu_{f2} - \mu_{b2}^2(3-4\mu_{b2})\mu_{f2}^2\} \\ & \quad +(b2 \leftrightarrow f2) \ , \\ P_W^{(3)}(1,\Lambda) &= -(16/3) \ \times (\mu_{b2}-\mu_{f2}) \\ & \quad \{1-\mu_{b2}(1+3\mu_{b2}) + 2\mu_{b2}^2\mu_{f2} + \mu_{b2}^2\mu_{f2}^2 + (b2 \leftrightarrow f2)\} \ , \\ P_W^{(4)}(1,\Lambda) &= (8/3) \ \times \\ & \quad \{\mu_{b2}(2-\mu_{b2}-3\mu_{b2}^2) - \mu_{b2}(6-3\mu_{b2}-10\mu_{b2}^2)\mu_{f2} - \mu_{b2}^2(8\mu_{b2}-3)\mu_{f2}^2\} \\ & \quad +(b2 \leftrightarrow f2) \ , \end{aligned} \tag{121}$$

the boundary contributions of the first two source components compensate each other,

$$c_W^{(2)}(1,M) = -\frac{8M^3}{(M^2-1)^2} = -\, c_W^{(3)}(1,M), \tag{122}$$

and the coefficient has the value

$$c(1,M) = c_W^{(4)}(1,M) = -\frac{4M^3}{(M^2-1)^2} \cdot \tag{123}$$

We finally turn to the term of second order in t. Here, both $c_V(2,M)$ and $c_W(2,M)$ contribute because $K_V^{(p)}(i_1,j_1,i_2,j_2|k_1,l_1,k_2,l_2)$ does not vanish. The polynomials $P_V^{(k)}(2,\Lambda)$ and $P_W^{(k)}(2,\Lambda)$ are given by

$$\begin{aligned}
P_V^{(2)}(2,\Lambda) = (128/9)\ \times& \\
\{\,[\,8\mu_{b1}\mu_{b2}\,&(1-2(\mu_{b1}+\mu_{b2})+4\mu_{b1}\mu_{b2}+\mu_{b1}^2+\mu_{b2}^2\,) \\
&+8(\mu_{b1}^2+\mu_{b2}^2)-16(\mu_{b1}^3+\mu_{b2}^3)+12(\mu_{b1}^4+\mu_{b2}^4)\,]\,\mu_{f1}\mu_{f2} \\
-[\,\mu_{b1}\mu_{b2}(\,&7(\mu_{b1}+\mu_{b2})-4(\mu_{b1}\mu_{b2}+\mu_{b1}^2+\mu_{b2}^2)\,) \\
&-4(\mu_{b1}^3+\mu_{b2}^3)\,]\,\mu_{f1}\mu_{f2}(\mu_{f1}+\mu_{f2}) \\
-4\mu_{b1}\mu_{b2}[\,&2(\mu_{b1}+\mu_{b2})-3\mu_{b1}\mu_{b2}\,]\,(\mu_{f1}^2+\mu_{f2}^2)+36\mu_{b1}^2\mu_{b2}^2\,\mu_{f1}^2\mu_{f2}^2 \\
+(b1\leftrightarrow f1,\ &b2\leftrightarrow f2)\,\}\ , \\
P_W^{(2)}(2,\Lambda) = -(64/45)(&\mu_{b2}-\mu_{f2})\times \\
\{\,7-52\mu_{b2}+&70\mu_{b2}^2-100\mu_{b2}^3+90\mu_{b2}^4 \\
&-2\mu_{b2}(\,15-70\mu_{b2}+70\mu_{b2}^2-42\mu_{b2}^3\,)\,\mu_{f2} \\
-\mu_{b2}^2(\,225&-404\mu_{b2}+162\mu_{b2}^2\,)\,\mu_{f2}^2-86\mu_{b2}^3\,\mu_{f2}^3+(b2\leftrightarrow f2)\,\}\ , \\
P_V^{(3)}(2,\Lambda) = (512/9)&\times \\
\{\,[\,2\mu_{b1}\mu_{b2}(&\mu_{b1}^2+\mu_{b2}^2+\mu_{b1}\mu_{b2})+2(\mu_{b1}^2+\mu_{b2}^2) \\
&-4(\mu_{b1}^3+\mu_{b2}^3)+3(\mu_{b1}^4+\mu_{b2}^4\,)]\,\mu_{f1}\mu_{f2} \\
+[\,\mu_{b1}\mu_{b2}(\,&\mu_{b1}^2+\mu_{b2}^2-8\mu_{b1}\mu_{b2}\,)+\mu_{b1}^3+\mu_{b2}^3\,]\,\mu_{f1}\mu_{f2}(\mu_{f1}+\mu_{f2}) \\
+\mu_{b1}\mu_{b2}[\,&2(\mu_{b1}+\mu_{b2})-3\mu_{b1}\mu_{b2}]\,(\mu_{f1}^2+\mu_{f2}^2) \\
-(b1\leftrightarrow f1,\ &b2\leftrightarrow f2)\,\}\ , \\
P_W^{(3)}(2,\Lambda) = (256/45)&\times \\
\{\,\mu_{b2}(7-26&\mu_{b2}+35\mu_{b2}^2-50\mu_{b2}^3+45\mu_{b2}^4) \\
+\mu_{b2}(5-35&\mu_{b2}+10\mu_{b2}^2+25\mu_{b2}^3-42\mu_{b2}^4)\mu_{f2} \\
+\mu_{b2}^2(90-&115\mu_{b2}+20\mu_{b2}^2-9\mu_{b2}^3)\,\mu_{f2}^2 \\
+5\mu_{b2}^3(5+&3\mu_{b2})\,\mu_{f2}^3+(b2\leftrightarrow f2)\,\}\ , \\
P_V^{(4)}(2,\Lambda) = P_V^{(2)}&(2,\Lambda)\ ,
\end{aligned}$$

$$
\begin{aligned}
P_W^{(4)}(2,\Lambda) = &-(2/45)(\mu_{b2}-\mu_{f2})\times \\
&\{\,7-52\mu_{b2}+70\mu_{b2}^2-100\mu_{b2}^3+90\mu_{b2}^4 \\
&+2\mu_{b2}(75-230\mu_{b2}+230\mu_{b2}^2-138\mu_{b2}^3)\,\mu_{f2} \\
&+\mu_{b2}^2(435-676\mu_{b2}+198\mu_{b2}^2)\,\mu_{f2}^2 \\
&+154\mu_{b2}^3\,\mu_{f2}^3+(b2\leftrightarrow f2)\,\}\;.
\end{aligned} \tag{124}
$$

The contributions of the three source components to $c_\Omega(2,M)$ are

$$
\begin{aligned}
c_V^{(2)}(2,M) &= 2\mathcal{N}(M^6-9M^4+18M^2+18)\;, \\
c_W^{(2)}(2,M) &= 4\mathcal{N}(4M^6-12M^4+45M^2-9)\;, \\
\\
c_V^{(3)}(2,M) &= 2\mathcal{N}M^2(11M^2-39)\;, \\
c_W^{(3)}(2,M) &= -2\mathcal{N}M^2(9M^4-22M^2+69)\;, \\
\\
c_V^{(4)}(2,M) &= \mathcal{N}M^2(M^6-9M^4+18M^2+18)\;, \\
c_W^{(4)}(2,M) &= -\mathcal{N}M^2(M^6-18M^4+51M^2-90)\;,
\end{aligned} \tag{125}
$$

where

$$
\mathcal{N} = 16/(3(M^2-1)^2(M^2-4)(M^2-9))\;. \tag{126}
$$

Similarly as in the terms of zeroth and first order, the total contribution of the source components $R_2(Q)$ and $R_3(Q)$ is equal to zero, $c^{(2)}(2,M)+c^{(3)}(2,M)=0$, and we find that

$$
c(2,M) = c_V^{(4)}(2,M)+c_W^{(4)}(2,M) = \frac{16M^2(3M^4-11M^2+36)}{(M^2-1)^2(M^2-4)(M^2-9)}\;\cdot \tag{127}
$$

Collecting the terms of all three orders, we get the leading part $C_{AE}(t,M)$ of the asymptotic expansion (88) in the form

$$
\begin{aligned}
C_{AE}(t,M) = &\frac{M^2}{4(M^2-1)}-\frac{4M^3}{(M^2-1)^2}\,t \\
&+\frac{16M^2(3M^4-11M^2+36)}{(M^2-1)^2(M^2-4)(M^2-9)}\,t^2.
\end{aligned} \tag{128}
$$

(For $M=2$, only the first two terms are meaningful.) This is our final analytic result.

3.3 Comparison with the Lorentzian form

Efetov [9] has shown that in the limit $M \gg 1$, the autocorrelation function has the form of the square of a Lorentzian,

$$C_{SL}(t, M) = C(0, M) \left[\frac{1}{1 + (t/t_d)}\right]^2 , \tag{129}$$

where t_d denotes a decay (shape) parameter. Thus, it is of interest to compare our analytical result (128) with the Taylor expansion of the squared Lorentzian at $t = 0$. It is given by

$$C_{SL}(t, M) = \frac{M^2}{4(M^2 - 1)} \left\{1 - 2(\frac{t}{t_d}) + 3(\frac{t}{t_d})^2 + \dots\right\} . \tag{130}$$

For $M \gg 1$, the two linear terms become identical if we put $t_d = M/8$, and the quadratic terms then coincide in this limit. This confirms the expected agreement for $M \gg 1$. However, for the small channel numbers M typical for the experiments mentioned in the Introduction, no choice of t_d exists for wich the linear and quadratic terms in the expansion of the squared Lorentzian would agree with the form (128). Put differently, with $t_d = M/8$, the decrease of the squared Lorentzian near $t = 0$ is much smaller than that of the autocorrelation function (128). A numerical test for the asymptotic expansion (128) is presented in the next Section.

4 Numerical Results

To allow for a comparison between the squared Lorentzian and the autocorrelation function of our random matrix model also for larger values of t but small M, we have calculated the latter function numerically. Here we present only the main results of this study. A more detailed presentation of the numerical method and the results will be published elsewhere.

After rescaling and the introduction of the now more convenient parameter $b = k(B^{(1)} - B^{(2)})A/\phi_0$, $\quad b^2 = 4t$, the quantity of interest takes the form (cf. Subsection 1.1)

$$\overline{g^{(1)}g^{(2)}} = \int \mathcal{D}[H^{(I)}]\mathcal{D}[H^{(II)}] \exp\left\{-\frac{N}{2}\left[\mathrm{tr}[H^{(I)}]^2 + \mathrm{tr}[H^{(II)}]^2\right]\right\} g^{(1)}g^{(2)} . \tag{131}$$

The conductances $g^{(r)}$ are given in terms of the S matrices $S^{(r)}$. The latter have the form

$$
\begin{aligned}
&S^{(r)} = 1 - 2i\pi W[D^{(r)}]^{-1}W^T \ , && D^{(r)} = -H^{(r)} + iW^T W \ , \\
&H^{(1)} = H^{(I)} - \frac{b}{2\sqrt{N}} H^{(II)} \ , && H^{(2)} = H^{(I)} + \frac{b}{2\sqrt{N}} H^{(II)} \ , \\
&W_{c\mu} = \delta_{c\mu} \ , \qquad \overline{H^{(C)}_{\mu\nu} H^{(C)}_{\mu'\nu'}} = \frac{1}{N}\delta_{\mu\nu'}\delta_{\mu'\nu} \ , && C = I, II \ .
\end{aligned}
\tag{132}
$$

The ensemble average was performed by repeated drawings of the random matrices $H^{(I)}, H^{(II)}$ from a random number generator. The results for $M = 2, 4$ and 10, calculated with matrices of dimension $N = 50$, are shown in Fig. 1. The solid line shows the best fit by a squared Lorentzian written in the form

$$
C_{SL}(b, M) = C(0, M) \left[\frac{1}{1 + (b/b_d)^2} \right]^2 \ , \tag{133}
$$

with $C(0, M) = M^2/[4(M^2 - 1)]$. The values of the shape (decay) parameter b_d are presented in Table 1. For comparison, the table gives also the values of b_d suggested by the results of Efetov [9] and Frahm [10].

The table shows that the best values for b_d are very close to Efetov's value $\sqrt{M/2}$. For all values of M considered, the autocorrelation function $C(b, M)$ agrees rather well with the squared Lorentzian. As expected, the agreement improves with increasing M. For $M = 2$, the deviations of $C(b, M)$ from the best squared–Lorentzian fit $C_{SL}(b, M)$ (with $b_d = 1.1$) are shown in more detail in Figs. 2 and 3. Even in this case, the deviations do not exceed 5% in magnitude. For $M > 2$, the best fit curve for $C_{SL}(b, M)$ lies within the error bars of the numerical calculation.

As a test of the asymptotic expansion (128), we have calculated numerically also the second derivative $\partial^2 C(b, M)/\partial b^2$ at $b = 0$ for the two lowest M values, $M = 2$ and $M = 4$. We assumed that taking the derivatives can be interchanged with taking the average. The derivatives of the integrand in Eq. (131) were analytically done by applying to the resolvents $D^{(r)}$ the formula

$$
\frac{\partial^n}{\partial x^n} \frac{1}{A + xB} = n!(-)^n \frac{1}{A + xB} \left(B \frac{1}{A + xB} \right)^n \ , \tag{134}
$$

with A, B denoting matrices independent of x. Table 2 compares the second derivatives with the values derived from the asymptotic expansion, and with

the corresponding derivatives of $C_{SL}(b, M)$. Expressed in terms of b, the expansion takes the form

$$
\begin{aligned}
C_{AE}(b, M) = & \frac{M^2}{4(M^2-1)} - \frac{M^3}{(M^2-1)^2}\, b^2 \\
& + \frac{M^2(3M^4 - 11M^2 + 36)}{(M^2-1)^2(M^2-4)(M^2-9)}\, b^4 .
\end{aligned} \tag{135}
$$

The values derived from the asymptotic expansion agree very well with the numerical results for both $M = 2$ and $M = 4$. This is not true for the derivative of the squared Lorentzian which for $M = 2$ differs considerably from the numerical result.

The same calculation of the fourth derivative $\partial^4 C(0,2)/\partial b^4$ yielded a divergent result. This fact caused us to analyse the second derivative $\partial^2 C(b,2)/\partial b^2$ at small $|b|$ in greater detail. The result is shown in Fig. 4. In contrast to the second derivative of the squared Lorentzian, which at the origin rises with the second power of b, the second derivative of $C(b, 2)$ seems to rise with the absolute value of b. The derivatives of this term linear in $|b|$ then generates a δ–function singularity in the fourth derivative. In the power spectrum of $C(b, 2)$, i.e. in the Fourier transform, such a singularity manifests itself in an algebraic decay for large k.

In summary we have shown that at the 5% level of accuracy, there is no difference between our results and a squared Lorentzian even for the smallest M value, $M = 2$. Closer inspection of the autocorrelation function for $M = 2$ and at the point $b = 0$ shows, however, that there is strong evidence for a singularity of the fourth derivative, caused by non–analytic behavior. This is an interesting and unexpected result for which we have no physical explanation at present.

5 Summary and Conclusions

We have investigated the magnetoconductance autocorrelation function for ballistic electron transport through microstructures having the form of a classically chaotic billiard. The structures were assumed to be connected to ideal leads carrying few channels. Assuming ideal coupling between leads and billiard, we have described this system in terms of a random matrix model.

The autocorrelation function depends only on the field parameter t, specified by the field (flux) difference, and on the channel number M. Using the supersymmetry technique, we have analytically calculated the leading terms of the asymptotic expansion of the correlation function at small t. To this end, we have used integral theorems obtained by applying Berezin's method of boundary functions. Using a generalization of the standard polar coordinates to parametrize the coset space, we succeeded in identifying and evaluating both volume and boundary (or Efetov–Wegner) terms. We believe that the method developed in this paper is of general interest for the supersymmetry technique, and we hope that it will be helpful in other cases.

We have shown that the first two terms of the asymptotic expansion of the autocorrelation function are entirely given by the Efetov–Wegner terms. This result is likely to be of general importance: Given some seemingly very natural choice of integration variables, the boundary terms may easily yield a or the major contribution.

For large M, the asymptotic expansion agrees with the corresponding small t expansion of the squared Lorentzian suggested by semiclassical theory. For small M, differences exist. These were studied further by combining our analytical work with numerical simulations. For the smallest value of M, $M = 2$, the difference between the autocorrelation function and the best squared–Lorentzian fit exists but does not exceed 5% in magnitude. This may seem irrelevant. However, a study of the derivatives of the autocorrelation function yielded further evidence for a statement suggested by the analytical form of the asymptotic expansion: The autocorrelation function seems to be non–analytic in t at $t = 0$. We find this result surprising. It suggests the occurrence of non–analytic behavior also in other correlation functions where the consequences may even be observable.

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6 Appendices

6.1 Matrices and Measures

In the volume integrals $\mathcal{I}_V$, the matrices U nad Λ have the form

$$U = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \cos\frac{1}{2}\theta & i\sin\frac{1}{2}\theta \\ i\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta \end{pmatrix}, \tag{136}$$

where θ denotes the diagonal matrix $\theta = \mathrm{diag}(i\vartheta_{b1}, i\vartheta_{b2}, \vartheta_{f1}, \vartheta_{f2})$, and where u_1 and u_2 denote the matrices

$$u_1 = u_{1x}u_{1\beta}u_{1\gamma}u_\phi, \qquad u_2 = u_{2x}u_{2\beta}u_{2\gamma}, \tag{137}$$

obtained by multiplying the factors (we set $x_{p\alpha} = \sin\zeta_{p\alpha}\mathrm{e}^{i\eta_{p\alpha}}$)

$$\begin{aligned}
u_{px} &= \begin{pmatrix} \cos\zeta_{pb} & \sin\zeta_{pb}\mathrm{e}^{i\eta_{pb}} & 0 & 0 \\ -\sin\zeta_{pb}\mathrm{e}^{-i\eta_{pb}} & \cos\zeta_{pb} & 0 & 0 \\ 0 & 0 & \cos\zeta_{pf} & \sin\zeta_{pf}\mathrm{e}^{i\eta_{pf}} \\ 0 & 0 & -\sin\zeta_{pf}\mathrm{e}^{-i\eta_{pf}} & \cos\zeta_{pf} \end{pmatrix}, \\
u_{1\beta} &= \begin{pmatrix} 1+\frac{1}{2}\beta_{11}\beta_{11}^\star & 0 & 0 & \beta_{11} \\ 0 & 1+\frac{1}{2}\beta_{12}\beta_{12}^\star & \beta_{12} & 0 \\ 0 & \beta_{12}^\star & 1-\frac{1}{2}\beta_{12}\beta_{12}^\star & 0 \\ \beta_{11}^\star & 0 & 0 & 1-\frac{1}{2}\beta_{11}\beta_{11}^\star \end{pmatrix}, \\
u_{2\beta} &= \begin{pmatrix} 1-\frac{1}{2}\beta_{21}\beta_{21}^\star & 0 & 0 & i\beta_{21} \\ 0 & 1-\frac{1}{2}\beta_{22}\beta_{22}^\star & i\beta_{22} & 0 \\ 0 & i\beta_{22}^\star & 1+\frac{1}{2}\beta_{22}\beta_{22}^\star & 0 \\ i\beta_{21}^\star & 0 & 0 & 1+\frac{1}{2}\beta_{21}\beta_{21}^\star \end{pmatrix}, \\
u_{1\gamma} &= \begin{pmatrix} 1+\frac{1}{2}\gamma_{11}\gamma_{11}^\star & 0 & \gamma_{11} & 0 \\ 0 & 1+\frac{1}{2}\gamma_{12}\gamma_{12}^\star & 0 & \gamma_{12} \\ \gamma_{11}^\star & 0 & 1-\frac{1}{2}\gamma_{11}\gamma_{11}^\star & 0 \\ 0 & \gamma_{12}^\star & 0 & 1-\frac{1}{2}\gamma_{12}\gamma_{12}^\star \end{pmatrix}, \\
u_{2\gamma} &= \begin{pmatrix} 1-\frac{1}{2}\gamma_{21}\gamma_{21}^\star & 0 & i\gamma_{21} & 0 \\ 0 & 1-\frac{1}{2}\gamma_{22}\gamma_{22}^\star & 0 & i\gamma_{22} \\ i\gamma_{21}^\star & 0 & 1+\frac{1}{2}\gamma_{21}\gamma_{21}^\star & 0 \\ 0 & i\gamma_{22}^\star & 0 & 1+\frac{1}{2}\gamma_{22}\gamma_{22}^\star \end{pmatrix},
\end{aligned} \tag{138}$$

and $u_\phi = \mathrm{e}^{i\phi}$, with $\phi = \mathrm{diag}(\phi_{b1}, \phi_{b2}, \phi_{f1}, \phi_{f2})$. The corresponding integration measure is

$$\mathrm{d}\mu(U\Lambda U^{-1}) = \mathrm{d}\mu(U)\mathrm{d}\mu(\Lambda) \ , \tag{139}$$

where $\mathrm{d}\mu(U)$ denotes the measure for integration over the matrices U given by

$$\begin{aligned}
\mathrm{d}\mu(U) &= \prod_p \mathrm{d}\mu(u_p) \ , \\
\mathrm{d}\mu(u_1) &= \mathrm{d}\mu(u_{1x})\mathrm{d}\mu(u_{1\beta})\mathrm{d}\mu(u_{1\gamma})\mathrm{d}\mu(\phi), \\
\mathrm{d}\mu(u_2) &= \mathrm{d}\mu(u_{2x})\mathrm{d}\mu(u_{2\beta})\mathrm{d}\mu(u_{2\gamma}) \ , \\
\mathrm{d}\mu(u_{px}) &= \prod_\alpha \mathrm{d}\zeta_{p\alpha}\mathrm{d}\eta_{p\alpha} \sin 2\zeta_{p\alpha} \ , \\
\mathrm{d}\mu(u_\phi) &= \prod_{\alpha r} \mathrm{d}\phi_{\alpha r} \ , \\
\mathrm{d}\mu(u_{p\beta}) &= \prod_q \mathrm{d}\beta_{pq}\mathrm{d}\beta^\star_{pq} \cdot \prod_p (1 - 2\beta_{p1}\beta^\star_{p1}\beta_{p2}\beta^\star_{p2}) \ , \\
\mathrm{d}\mu(u_{p\gamma}) &= \prod_r \mathrm{d}\gamma_{pr}\mathrm{d}\gamma^\star_{pr} \ ,
\end{aligned} \tag{140}$$

and where $\mathrm{d}\mu(\Lambda)$ denotes the measure for integration over the matrices Λ,

$$\mathrm{d}\mu(\Lambda) = \prod_{\alpha r} \mathrm{d}\mu_{\alpha r} \cdot \prod_\alpha (\mu_{\alpha 1} - \mu_{\alpha 2})^2 \cdot \prod_{rr'} (\mu_{br} - \mu_{fr'})^{-2} \ , \tag{141}$$

with $\mu_{br} = \cosh^2 \frac{1}{2}\vartheta_{br}$ and $\mu_{fr} = \cos^2 \frac{1}{2}\vartheta_{fr}$. The domains of integration over (the ordinary parts of) $\zeta_{p\alpha}$ extend from 0 to $\pi/2$, those of integration over $\eta_{p\alpha}, \phi_{\alpha r}$ from 0 to 2π, and those of integration over μ_{br} and μ_{fr} from 1 to ∞ and 1 to 0, respectively, with $\mu_{\alpha 1} < \mu_{\alpha 2}$. However, as discussed in Subsection 3.1, calculating the volume integrals can be simplified drastically by using a modified parametrization where the matrices u_1 and u_2 are given by the products

$$u_1 = u_{1\gamma}u_{1\beta}u_{1x}u_\phi, \qquad u_2 = u_{2\gamma}u_{2\beta}u_{2x} \ . \tag{142}$$

The integration measure in the modified parametrization has the same form as in the old parametrization.

In the boundary integrals $\mathcal{I}_W$, all variables labelled by $r = 1$ are to be set equal to zero: the matrices u_{px} and $u_{p\beta}$ appearing in U have again the form shown in Eqs. (138), whereas the matrices $u_{1\gamma}, u_{2\gamma}, \phi$ and θ simplify to

$$u_{1\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\frac{1}{2}\gamma_{12}\gamma_{12}^\star & 0 & \gamma_{12} \\ 0 & 0 & 1 & 0 \\ 0 & \gamma_{12}^\star & 0 & 1-\frac{1}{2}\gamma_{12}\gamma_{12}^\star \end{pmatrix},$$

$$u_{2\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\frac{1}{2}\gamma_{22}\gamma_{22}^\star & 0 & i\gamma_{22} \\ 0 & 0 & 1 & 0 \\ 0 & i\gamma_{22}^\star & 0 & 1+\frac{1}{2}\gamma_{22}\gamma_{22}^\star \end{pmatrix},$$

$$\phi = \mathrm{diag}(0, \phi_{b2}, 0, \phi_{f2}) \ ,$$
$$\theta = \mathrm{diag}(0, i\vartheta_{b2}, 0, \vartheta_{f2}) \ . \tag{143}$$

In the integration measure $\mathrm{d}\mu(U\Lambda U^{-1})$, the measures for integration over the matrices $u_{p\gamma}, u_\phi$ and Λ simplify to

$$\mathrm{d}\mu(u_{p\gamma}) = \mathrm{d}\gamma_{p2}\mathrm{d}\gamma_{p2}^\star \ , \qquad \mathrm{d}\mu(u_\phi) = \prod_\alpha \mathrm{d}\phi_{\alpha 2} \ ,$$
$$\mathrm{d}\mu(\Lambda) = \prod_\alpha \mathrm{d}\mu_{\alpha 2} \cdot \ (\mu_{b2} - \mu_{f2})^{-2} \ . \tag{144}$$

6.2 Integration over Eigenvalues

The integration over Λ of $F_W^{(k)}(n, \Lambda, M)$ leads to the two-dimensional integrals

$$\mathcal{I}(p2, q2) = \int_1^\infty \mathrm{d}\mu_{b2} \int_0^1 \mathrm{d}\mu_{f2} \frac{\mu_{f2}^{M-2+q2}}{\mu_{b2}^{M+2-p2}} \frac{1}{\mu_{b2} - \mu_{f2}} \ , \tag{145}$$

with $p2, q2$ nonnegative integers. The integrals converge if $M + 2 > p2$ and $M + q2 > 1$; since $M \geq 2$ for physical reasons, the second condition is always satisfied. With $x = \mu_{f2}/\mu_{b2}$ and $y = \mu_{f2}$, we can write

$$\mathcal{I}(p2, q2) = \int_0^1 \mathrm{d}x \ \frac{x^{M+1-p2}}{1-x} \int_x^1 \mathrm{d}y \ y^{p2+q2-4} \ . \tag{146}$$

For $p2 + q2 \neq 3$, the integration over y gives

$$\begin{aligned} \mathcal{I}(p2, q2) &= \frac{1}{3-p2-q2} \int_0^1 \mathrm{d}x \ \frac{x^{M-2+q2} - x^{M+1-p2}}{1-x} \\ &= \frac{1}{3-p2-q2} \left(\psi(M+2-p2) - \psi(M-1+q2)\right) \ , \end{aligned} \tag{147}$$

where $\psi(z)$ denotes Euler's ψ function [21]. Making use of

$$\psi(p+1) = \sum_{k=1}^{p} \frac{1}{k} + \psi(1) \tag{148}$$

yields

$$\begin{aligned}
\mathcal{I}(p2,q2) &= \frac{1}{3-p2-q2} \sum_{k=M-1+q2}^{M+1-p2} \frac{1}{k} \qquad \text{for} \quad p2+q2<3 \; , \\
\mathcal{I}(p2,q2) &= \frac{1}{p2+q2-3} \sum_{k=M+2-p2}^{M-2+q2} \frac{1}{k} \qquad \text{for} \quad p2+q2>3 \; .
\end{aligned} \tag{149}$$

For $p2+q2=3$, the integration over y leads to

$$\mathcal{I}(p2,3-p2) = -\int_0^1 \mathrm{d}x \frac{x^{M+1-p2}\ln x}{1-x} = \zeta(2, M+2-p2) \; , \tag{150}$$

where $\zeta(z,q)$ denotes Riemann's ζ function [22]. Substituting the explicit expression, we get

$$\mathcal{I}(p2,3-p2) = \frac{\pi^2}{6} - \sum_{k=1}^{M+1-p2} \frac{1}{k^2} \; . \tag{151}$$

The integration over Λ of $F_V^{(k)}(n,\Lambda,M)$ leads to the four-dimensional integrals

$$\begin{aligned}
&\mathcal{I}(p1,p2,q1,q2) \\
&= \int_1^\infty \mathrm{d}\mu_{b1} \int_1^\infty \mathrm{d}\mu_{b2} \int_0^1 \mathrm{d}\mu_{f1} \int_0^1 \mathrm{d}\mu_{f2} \; \frac{\mu_{f1}^{M-2+q1}\mu_{f2}^{M-2+q2}}{\mu_{b1}^{M+2-p1}\mu_{b2}^{M+2-p2}} \; \frac{\prod_\alpha \; (\mu_{\alpha 1}-\mu_{\alpha 2})^2}{\prod_{rr'}(\mu_{br}-\mu_{fr'})} \; ,
\end{aligned} \tag{152}$$

with $p1,p2,q1,q2$ nonnegative integers. Using the identity

$$\begin{aligned}
\frac{\prod_\alpha \; (\mu_{\alpha 1}-\mu_{\alpha 2})^2}{\prod_{rr'}(\mu_{br}-\mu_{fr'})} &= -\frac{\mu_{b1}-\mu_{b2}}{\mu_{b1}-\mu_{f1}} - \frac{\mu_{b1}-\mu_{b2}}{\mu_{b1}-\mu_{f2}} + \frac{\mu_{b1}-\mu_{b2}}{\mu_{b2}-\mu_{f1}} + \frac{\mu_{b1}-\mu_{b2}}{\mu_{b2}-\mu_{f2}} \\
&\quad - \frac{(\mu_{b1}-\mu_{b2})^2}{(\mu_{b1}-\mu_{f1})(\mu_{b2}-\mu_{f2})} - \frac{(\mu_{b1}-\mu_{b2})^2}{(\mu_{b1}-\mu_{f2})(\mu_{b2}-\mu_{f1})} \; ,
\end{aligned} \tag{153}$$

we can express this integral in terms of the two-dimensional integrals (145) just considered, as seen from

$$\int_1^\infty \mathrm{d}\mu_{b1} \int_1^\infty \mathrm{d}\mu_{b2} \int_0^1 \mathrm{d}\mu_{f1} \int_0^1 \mathrm{d}\mu_{f2} \, \frac{\mu_{f1}^{M-2+q1} \mu_{f2}^{M-2+q2}}{\mu_{b1}^{M+2-p1} \mu_{b2}^{M+2-p2}} \frac{\mu_{b1}-\mu_{b2}}{\mu_{b1}-\mu_{f1}}$$
$$= \frac{1}{(M+1-p2)(M-1+q2)} \mathcal{I}(p1+1,q1)$$
$$- \frac{1}{(M-p2)(M-1+q2)} \mathcal{I}(p1,q1) \, , \tag{154}$$

and

$$\int_1^\infty \mathrm{d}\mu_{b1} \int_1^\infty \mathrm{d}\mu_{b2} \int_0^1 \mathrm{d}\mu_{f1} \int_0^1 \mathrm{d}\mu_{f2} \, \frac{\mu_{f1}^{M-2+q1} \mu_{f2}^{M-2+q2}}{\mu_{b1}^{M+2-p1} \mu_{b2}^{M+2-p2}} \frac{(\mu_{b1}-\mu_{b2})^2}{(\mu_{b1}-\mu_{f1})(\mu_{b2}-\mu_{f2})}$$
$$= \mathcal{I}(p1+2,q1) \, \mathcal{I}(p2,q2) - 2\mathcal{I}(p1+1,q1) \, \mathcal{I}(p2+1,q2)$$
$$+ \mathcal{I}(p1,q1) \, \mathcal{I}(p2+2,q2) \, . \tag{155}$$

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Table 1: The value of the shape parameter b_d for the best squared-Lorentzian fit $C_{SL}(b, M)$ for the channel numbers $M = 2, 4$ and 10. The last two columns present the b_d values $\sqrt{M/2}$ and $\sqrt{(M+1)/2}$ predicted by Efetov [9] and Frahm [10] respectively.

M	b_d	$\sqrt{M/2}$	$\sqrt{(M+1)/2}$
2	1.10	1.00	1.22
4	1.44	1.41	1.58
10	2.20	2.24	2.35

Table 2: The value of the second derivative of the correlation function $C(b, M)$ at $b = 0$ for $M = 2$ and $M = 4$. The values obtained from the asymptotic expansion $C_{AE}(b, M)$ and from the best squared-Lorentzian fit $C_{SL}(b, M)$ are shown for comparison.

M	$\partial^2 C(0, M)/\partial b^2$	$\partial^2 C_{AE}(0, M)/\partial b^2$	$\partial^2 C_{SL}(0, M)/\partial b^2$
2	-1.76	-1.78	-1.08
4	-0.52	-0.57	-0.50

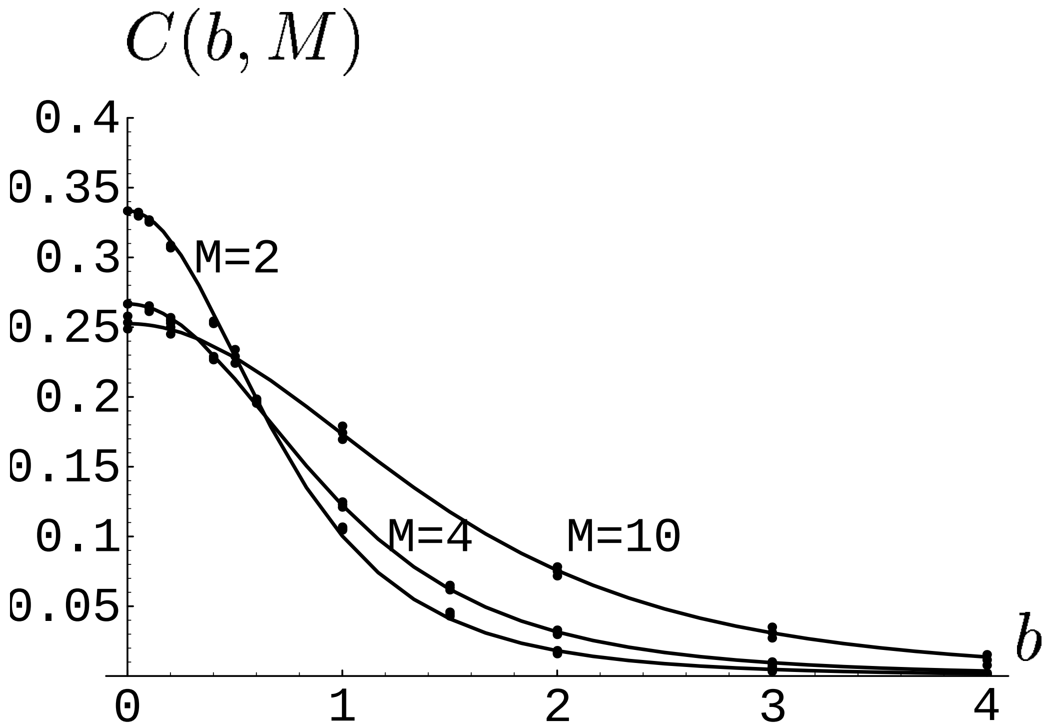

Figure 1: The correlation function $C(b, M)$ as a function of the magnetic-field parameter b for the channel numbers $M = 2, 4$ and 10. The solid lines show the corresponding best squared–Lorentzian fit $C_{SL}(b, M)$. The calculated values of the correlation function are lying in the middle of the error bar intervals.

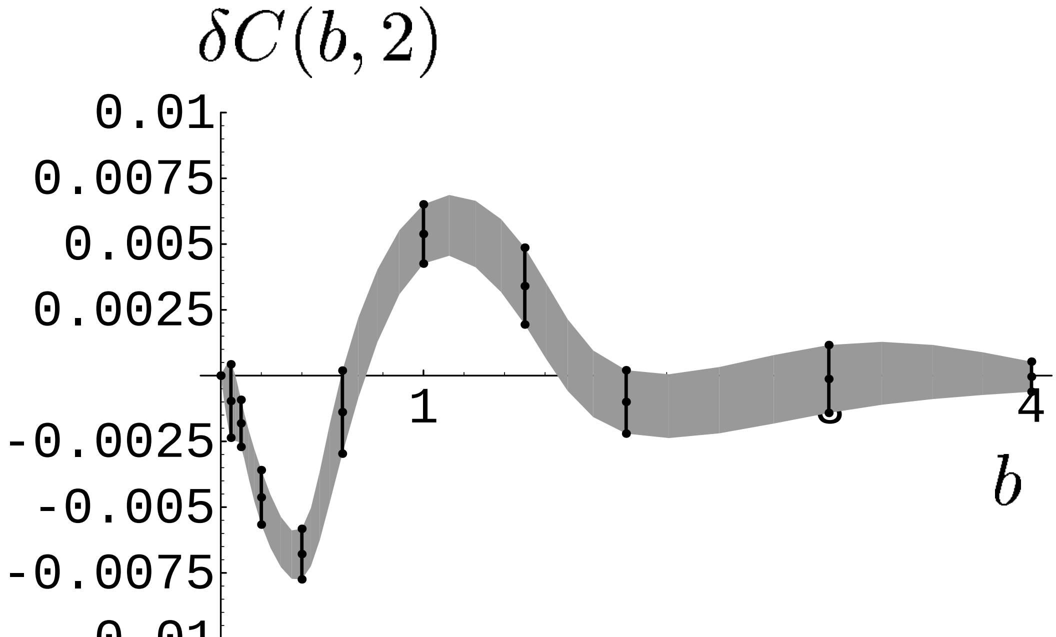

Figure 2: The deviation $\delta C(b,2) = C(b,2) - C_{SL}(b,2)$ of the correlation function $C(b,2)$ from its best squared–Lorentzian fit $C_{SL}(b,2)$ as a function of b. The gray band connecting the error bars is intended to help the eye.

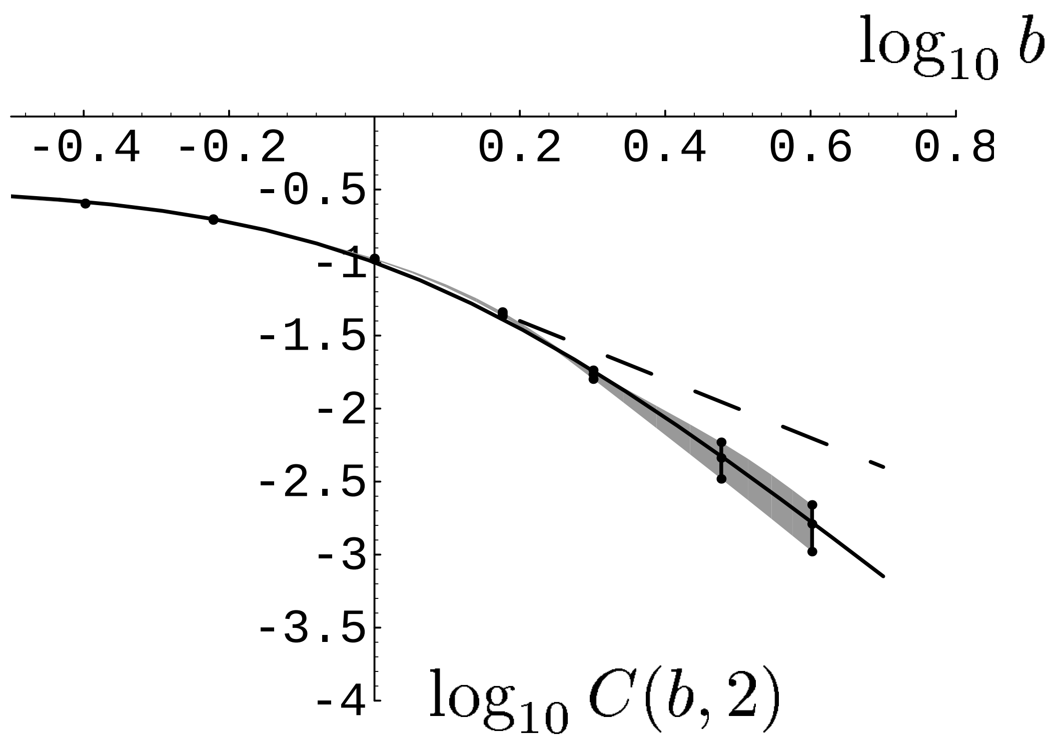

Figure 3: Plot of $\log_{10} C(b, 2)$ as a function of $\log_{10} b$. The solid line refers to the best squared–Lorentzian fit, the dashed line to the b^{-2} law, for comparison.

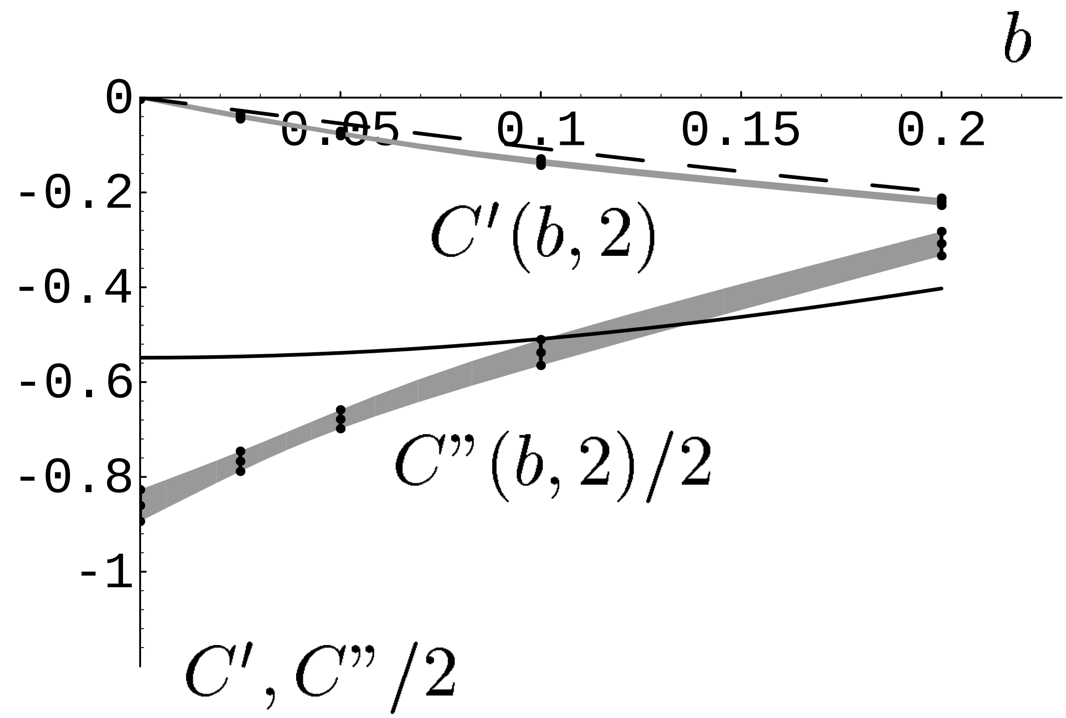

Figure 4: Graph of the first two derivatives of $C(b,2)$ as a function of b. The upper curve shows the derivative $C'(b,2) = \partial C(b,2)/\partial b$, the lower curve the function $C''(b,2) = \partial^2 C(b,2)/\partial b^2$ scaled by the factor 1/2. The dashed and the solid line show their respective counterparts for the best squared–Lorentzian fit $C_{SL}(b,2)$.